Super-multiplicativity and a lower bound for the decay of the signature of a path of finite length

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Abstract

For a path of length $L > 0$, if for all $n \geq 1$, we multiply the n-th term of the signature by $n!L^{-n}$, we say the resulting signature is '*normalised*'. It has been established[3] that the norm of the n-th term of the normalised signature of a bounded-variation path is bounded above by 1. In this article we discuss the super-multiplicativity of the norm of the signature of a path with finite length, and prove by Fekete's lemma the existence of a non-zero limit of the n-th root of the norm of the n-th term in the normalised signature as n approaches infinity.

Résumé

Pour une trajectoire de longueur $L > 0$, si l'on multiplie le n-iéme terme de la signature par $n!L^{-n}$ pour tout $n \geq 1$, on la signature ainsi

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obtenue est dite "*normalisée*". Il a été établi en [3] que la norme du n-iéme terme de la signature normalisée d'une trajectoire à variation bornée est majorée par 1. Dans cet article nous étudions la super-multiplicativité de la norme de la signature d'une trajectoire de longueur finie, et nous dmontrons à l'aide du lemme de Fekete l'existence d'une limite non nulle lorsque n tend l'infini pour la racine n-iéme de la norme du n-iéme terme de la signature normalisée.

1 Super-multiplicativity of the signature in reasonable tensor algebra norms

Definition 1. *Let* $\{V_j\}_{j=1}^N$ *be normed vector spaces over* $\mathbb{F}$=$\mathbb{R}$ *or* $\mathbb{C}$. *Their* algebraic tensor product space *is defined as the vector space*

$$V_1 \otimes ... \otimes V_N = \left\{ \sum_{i \in I} v_i^1 \otimes ... \otimes v_i^N : v_i^j \in V_j, \quad \forall i \in I, |I| < \infty, J = 1, ..., N. \right\},$$

where we identify $(u + v) \otimes w = u \otimes w + v \otimes w$.

Definition 2. *If* $\phi_j \in V_j'$ *are bounded linear functionals on* V_j, $j = 1, ..., N$, *then we define the dual action of* $\phi_1 \otimes ... \otimes \phi_N$ *on* $V_1 \otimes ... \otimes V_N \to \mathbb{F}$ *by*

$$(\phi_1 \otimes ... \otimes \phi_N)(\sum_{i \in I} v_i^1 \otimes ... \otimes v_i^N) := \sum_{i \in I} \prod_{j=1}^N \phi(v_i^j)$$

for all $v_i^j \in V_j, j = 1, ..., N, i \in I$, $|I| < \infty$. *The map is well-defined and independent of the representation on the right-hand side.*

Now we state the properties of the norms on tensor products that are required for this article.

Definition 3 (Reasonable tensor algebra norm)**.** *Let* $V, V \otimes V, ..., V^{\otimes n}$ *be normed vector spaces. We assume that for all* $v \in V^{\otimes n}$, $w \in V^{\otimes m}$,

$$\|v \otimes w\| \leq \|v\| \|w\| \tag{1}$$

and the norm induced on the dual spaces satisfies that for all $\phi \in (V^{\otimes m})'$, $\psi \in (V^{\otimes n})'$,

$$\|\phi \otimes \psi\| \leq \|\phi\| \|\psi\|. \tag{2}$$

Moreover, if $S(n)$ *denotes the symmetric group over* $\{1, 2, .., n\}$, *we assume that for all* $n \geq 1$,

$$\|\sigma(v)\| = \|v\| \quad \forall \sigma \in S(n), \, v \in V^{\otimes n}.$$

Proposition 1 (Ryan[4]). *Let X and Y be normed vector spaces. If $\| \ \|$ is a tensor norm on $X \otimes Y$ which satisfies*

$$\|v \otimes w\| \leq \|v\|\|w\| \quad \forall v \in X, w \in Y;$$

and the norm induced on the dual spaces satisfies

$$\|\phi \otimes \psi\| \leq \|\phi\|\|\psi\| \quad \forall \phi \in X', \psi \in Y',$$

then $\| \ \|$ is called a reasonable cross norm*, and $\|x \otimes y\| = \|x\|\|y\|$ for every $x \in X$ and $y \in Y$; for every $\phi \in X'$ and $\psi \in Y'$, the norm of the linear functional $\phi \otimes \psi$ on $(X \otimes Y, \| \ \|)$ satisfies $\|\phi \otimes \psi\| = \|\psi\|\|\psi\|$.*

Using Proposition 1 implies that the inequalities in Equation (1) and (2) imply equality.

Remark 1. *Note that under the assumptions of Definition 3 for all $a \in V^{\otimes m}$, $b \in V^{\otimes n}$, $c \in V^{\otimes l}$,*

$$\|(a \otimes b) \otimes c\| = \|a \otimes (b \otimes c)\| = \|a\|\|b\|\|c\|.$$

We provide some examples of tensor norms which are reasonable tensor algebra norms.

Definition 4. *Let $\{V_j\}_{j=1}^N$ be normed vector spaces over $\mathbb{F}$. The* projective tensor norm *on $V_1 \otimes ... \otimes V_N$ is defined such that for $x \in V_1 \otimes ... \otimes V_N$,*

$$\|x\|_\pi := \inf \left\{ \sum_{i \in I} \|v_i^1\| ... \|v_i^N\| : x = \sum_{i \in I} v_i^1 \otimes ... \otimes v_i^N, v_i^j \in V_j \, \forall i \in I, |I| < \infty. \right\}.$$

The injective tensor norm *on $V_1 \otimes ... \otimes V_N$ is defined such that for $x = \sum_{i \in I} v_i^1 \otimes ... \otimes v_i^N \in V_1 \otimes ... \otimes V_N$, $i \in I$, $|I| < \infty$,*

$$\|x\|_\delta := \sup\{|\sum_{i \in I} \prod_{j=1}^N \phi_j(v_i^j)| : \phi_j \in V_j', \|\phi_j\| \leq 1 \, \forall j = 1, ..., N\}$$

for any representation of x.

Lemma 1. *The projective tensor norm and the injective tensor norm defined in Definition 4 both satisfy the properties stated in Definition 3. Moreover, if α is a reasonable cross norm on $X \otimes Y$, and $u \in X \otimes Y$, then*

$$\|x\|_\delta \leq \alpha(x) \leq \|x\|_\pi.$$

Furthermore, any reasonable tensor algebra norm is sandwiched between the injective and projective tensor norms.

The proof of Lemma 1 is omitted here.

Lemma 2. *The Hilbert-Schmidt norm is a reasonable tensor algebra norm.*

The proof of Lemma 2 is omitted here.

Definition 5. *Let $V, V \otimes V, ..., V^{\otimes n}$ be Banach completed spaces equipped with a reasonable tensor algebra norm compatible with the norm on V, and $\gamma : J \to V$ be a continuous path with finite length. The* signature *of γ is denoted by*

$$S = (1, S_1, S_2, ..., S_n, ...), \tag{3}$$

where for each $n \geq 1$, $S_n = \int_{u_1 < ... < u_n, u_1,,,.u_n \in J} \mathrm{d}\gamma_{u_1} \otimes ... \otimes \mathrm{d}\gamma_{u_n}$.

Remark 2. *Note that the n-th term of S lives in the completed Banach space $V^{\otimes n}$ whenever the algebraic tensor product is completed with a reasonable tensor algebra norm.*

From now on we will fix a Banach space V, a reasonable tensor algebra norm, and we will take $V^{\otimes n}$ to be the completion of the algebraic tensor product with respect to that reasonable tensor algebra norm.

Definition 6 (Shuffle product)**.** *The shuffle product is defined inductively to be bilinear, and so that*

$$u \otimes a \sqcup\!\sqcup v \otimes b := (u \sqcup\!\sqcup v \otimes b) \otimes a + (u \otimes a \sqcup\!\sqcup v) \otimes b$$

for any $a, b \in V$.

Definition 7 (Group-like elements)**.** *Define*

$$\tilde{T}((V)) := \left\{ (a_0, a_1, a_2, ...) : a_n \in V^{\otimes n} \, \forall n \geq 1, a_0 = 1 \right\}.$$

An element $\mathbf{a} \in \tilde{T}((V))$ is called group-like *if for all $\phi, \psi \in (\tilde{T}((V)))'$,*

$$\phi \sqcup\!\sqcup \psi(\mathbf{a}) = \phi(\mathbf{a})\psi(\mathbf{a}).$$

Theorem 1. *Suppose $\gamma : J \to V$ is a path of finite length. Then for $m, n \geq 0$, the signature of γ satisfies*

$$\|(m+n)! S_{m+n}\| \geq \|n! S_n\| \|m! S_m\| \quad \forall m, n \geq 0. \tag{4}$$

where $\| \, \|$ is any reasonable tensor algebra norm. $V^{\otimes 0}$ is defined to be $\mathbb{F}$, and $S_0 = 1$.

Proof. By Hahn-Banach Theorem, there exists $\phi_n \in (V^{\otimes n})'$, $\phi_m \in (V^{\otimes m})'$ such that $\|\phi_n\| = 1$, $\|\phi_m\| = 1$, and

$$\phi_n(S_n) = \|S_n\|,\ \phi_m(S_m) = \|S_m\|.$$

Equivalently, we can write

$$\phi_n(S) = \|S_n\|,\ \phi_m(S) = \|S_m\|,$$

where we define $\phi_k(x) = 0$ for $x \notin V^{\otimes k}$ for all $k \geq 0$. From [3] we know that S is group-like, hence

$$\phi_m \sqcup\!\sqcup\, \phi_n(S) = \phi_m(S)\phi_n(S) = \|S_m\|\|S_n\|.$$

Also,

$$\begin{aligned}\phi_m \sqcup\!\sqcup\, \phi_n(S_{m+n}) &= \sum_{\sigma\in\text{Shuffles}(m,n)} \sigma(\phi_m \otimes \phi_n)(S_{m+n}) \\ &= \sum_{\sigma\in\text{Shuffles}(m,n)} (\phi_m \otimes \phi_n)(\sigma^{-1}(S_{m+n})),\end{aligned}$$

so

$$|\phi_m \sqcup\!\sqcup\, \phi_n(S_{m+n})| \leq \#\text{shuffles}(m,n)\|\phi_m \otimes \phi_n\|\|S_{m+n}\|.$$

Note that $\#\text{shuffles}(m,n) = \frac{(m+n)!}{n!m!}$, and by Definition 3 we know that

$$\|\phi_m \otimes \phi_n\| \leq \|\phi_m\|\|\phi_n\| = 1.$$

Hence

$$\|(m+n)!S_{m+n}\| \geq \|n!S_n\|\|m!S_m\|$$

as expected. □

Corollary 1. *If $S_j = 0$, then $S_k = 0$ for $k = 1, ..., j$.*

Proof. The proof follows from Theorem 1. □

2 Limiting behaviour

We note the following lemma by Fekete[5].

Theorem 2 (Fekete's Lemma). *If a sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ satisfies the sub-additivity condition*

$$a_{m+n} \leq a_m + a_n \quad \forall m, n \in \mathbb{N},$$

Then

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\in\mathbb{N}} \frac{a_n}{n}.$$

Theorem 3 (Asymptotic behaviour of the signature). *If $\gamma : J \to V$ is a continuous tree-reduced path of finite length $L > 0$, then under any reasonable tensor algebra norm $\| \|$. there exists a non-zero limit $\tilde{L}$ such that*

$$\begin{aligned} &\lim_{n\to\infty} \|n!S_n\|^{1/n} \\ &= \sup_{k\geq 1} \|k!S_k\|^{1/k} \\ &= \tilde{L} > 0. \end{aligned}$$

Proof. By Theorem 1, we know that for all $m, n \geq 0$,

$$\|(m+n)!S_{m+n}\| \geq \|n!S_n\| \|m!S_m\|.$$

Taking logarithm gives

$$-\log(\|(m+n)!S_{m+n}\| \leq -\log(\|n!S_n\|) - \log(\|m!S_m\|.$$

So the function $f(n) := -\log(\|n!S_n\|/L^n)$ satisfies $f(m+n) \leq f(m) + f(n)$ for all $m, n \in \mathbb{N}$. Then by Fekete's lemma[5], $\frac{1}{n}\log(\|n!S_n\|)$ converges to $\sup_{k\geq 1}\log(\|k!S_k\|)/k$, hence $\|n!S_n\|^{1/n}$ converges to $\sup_{k\in\mathbb{N}}\|k!S_k\|^{1/k}$. Note by Hambly and Lyons[2], every path of finite length has a unique tree-reduced [1] version with the same signature, if the tree-reduced path is non-trivial then there will be at least one term in the signature of the path which is non-zero. Hence $\sup_{k\geq 1}\|k!S_k\|^{1/k}$ is non-zero. Therefore $\|n!S_n\|^{1/n}$ converges to a non-zero limit as n increases. □

1. Roughly speaking, a tree-reduced path is the a path where it does not go back on cancelling itself over any interval.

Corollary 2. *Let V be a Banach space. For any element*

$$\mathbf{a} = (a_0, a_1, a_2, ...) \in \left\{(b_0, b_1, b_2, ...) : b_0 = 1, b_n \in V^{\otimes n} \, \forall n \geq 1\right\}$$

which is group-like, we have

$$\|(m+n)!a_{m+n}\| \geq \|m!a_m\| \|n!a_n\| \quad \forall m, n \geq 0,$$

and $\|n!a_n\|^{1/n}$ converges to $\sup_{k\in\mathbb{N}} \|k!a_k\|^{1/k}$ as n increases under any reasonable tensor algebra norm $\| \ \|$.

Proof. Note that since $\mathbf{a}$ is group-like, the same arguments apply as in Theorem 1 and Theorem 3. □

Remark 3. *It is an interesting question to ask whether there is an nice and simple form of the limit of $\|n!S_n\|^{1/n}$ mentioned in Theorem 3, and whether the limit is the same under any reasonable tensor algebra norm. Moreover, we know from [3] that for a path with finite length $L > 0$, an upper bound of $\|n!S_n\|$ is L^n. Furthermore, Lyons and Hambly[2] proved that for a smooth enough path of finite length, the ratio $\|n!S_n\|/L^n$ converges to 1 under certain norms. Therefore we have the following conjecture.*

Conjecture 1. *Let V be a Banach space, and $\gamma : J \to V$ be a path with finite length $L > 0$. Then the signature of γ satisfies that*

$$\|n!S_n\|^{1/n} \to L \quad \text{as } n \to \infty,$$

under any reasonable tensor algebra norm .

Remark 4. *An interesting tensor norm to consider is the* Haagerup tensor norm*[1]. Clearly the Haagerup norm is not a reasonable tensor algebra norm, however under the Haagerup norm, for a path of finite length $L > 0$, we still have $n!\|S_n\| \leq L^n$. Therefore it is an interesting question to ask whether the signature will have the same behaviour as described in Theorem 3 under the Haagerup tensor norm, or the symmetrised forms of the Haagerup tensor norm.*

Remark 5. *Although it has been shown that $\|n!S_n\|$ eventually behaves like L^n under certain norms for well-behaved paths(see [2]), some simple examples show that in general for a path with finite length, $\|n!S_n\|/L^n$ does not necessarily converge to 1 as n increases. Therefore the result in Theorem 3 is the best description we can have about the decay of the signature for a path with finite length.*
For a p-variation path where $p > 1$, by considering simple examples we can see that we cannot have a non-zero limit for $\|(n/p)!S_n\|^{1/n}$ as n increases.

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References

[1] David P Blecher and Vern I Paulsen. Tensor products of operator spaces. *Journal of Functional Analysis*, 99(2):262–292, 1991.

[2] Ben Hambly and Terry Lyons. Uniqueness for the signature of a path of bounded variation and the reduced path group. *Annals of Mathematics*, pages 109–167, 2010.

[3] Terry J Lyons, Michael Caruana, and Thierry Lévy. *Differential equations driven by rough paths*. Springer, 2007.

[4] Raymond A Ryan. *Introduction to tensor products of Banach spaces*. Springer Science & Business Media, 2013.

[5] J Michael Steele. *Probability theory and combinatorial optimization*, volume 69. Siam, 1997.