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THE NUMÉRAIRE PROPERTY AND LONG-TERM GROWTH OPTIMALITY FOR DRAWDOWN-CONSTRAINED INVESTMENTS

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We consider the portfolio choice problem for a long-run investor in a general continuous semimartingale model. We combine the decision criterion of pathwise growth optimality with a flexible specification of attitude toward risk, encoded by a linear drawdown constraint imposed on admissible wealth processes. We define the constrained numéraire property through the notion of expected relative return and prove that drawdown-constrained numéraire portfolio exists and is unique, but may depend on the investment horizon. However, when sampled at the times of its maximum and asymptotically as the time-horizon becomes distant, the drawdown-constrained numéraire portfolio is given explicitly through a model-independent transformation of the unconstrained numéraire portfolio. The asymptotically growth-optimal strategy is obtained as limit of numéraire strategies on finite horizons.

KEY WORDS: drawdown constraints, numéraire property, asymptotic growth, portfolio risk management.

1. INTRODUCTION

1.1. Drawdown-Constrained Investment Problem

We consider an optimal investment problem under a drawdown constraint which stipulates that the wealth process never falls below a given fraction α of its past maximum. In particular, even the most adverse market crash may not reduce the investor's wealth by more than $100(1-\alpha)\%$. Such constraints mitigate investors' risk by effectively introducing a stop-loss safety trigger to avoid large drawdowns. They are commonly encountered in practice and are related to the way investments are assessed both by market participants as well as regulators. Performance measures involving drawdowns include, for example,

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Calmar ratio, Sterling ratio, and Burke ratio, see Eling and Schuhmacher (2007) and chapter 4 of Bacon (2008); see Lhabitant (2004) for a more detailed discussion on their practical use. It should also be noted that drawdowns are often reported; for example, the Commodity Futures Trading Commission's mandatory disclosure regime stipulates that managed futures advisers report their "worst peak-to-valley drawdown."

We investigate decision making based on optimality expressed through the *numéraire property* in the spirit of Long (1990). We require that expected relative returns of any other nonnegative investment with respect to the wealth generated by the optimal portfolio over the same time period are nonpositive. In fact, this choice of optimality arises in an axiomatic way from numéraire[1]-invariant preferences, as set forth in Kardaras (2010b). In the unconstrained case, the global numéraire portfolio $\widehat{X}$ is the wealth process which has the property that all other investments, denominated in units of $\widehat{X}$, are supermartingales. It is well known that $\widehat{X}$ also maximizes the asymptotic long-term growth-rate and is the investment corresponding to the Kelly's criterion (Kelly 1956)—see, for example, Hakansson (1971), Bansal and Lehmann (1997) and the references therein. Some recent contributions explored the numéraire property in a constrained investment universe. In particular, Karatzas and Kardaras (2007) showed that with pointwise convex constraints on the fractions invested in each asset, one can retrieve existence and all useful properties of the numéraire portfolio. We contribute to this direction of research by providing a detailed analysis of the numéraire property within the class of investments which satisfy a given linear drawdown constraint.

1.2. Main Results

We work in a general continuous-path semimartingale setup. Our first main result establishes existence of unique portfolios with the numéraire property over different time-horizons for drawdown-constrained investment. In contrast to the unconstrained case, the optimal strategies may depend on the time-horizon, which we demonstrate with an explicit general construction.

Our second main result considers a long-run investor. Given the investor's acceptable level of drawdown α, we show that there is a unique choice of investment strategy that almost surely asymptotically outperforms any other strategy which satisfies the α-drawdown constraint. The optimal strategy is given explicitly in two manners. First, we obtain a version of the mutual fund theorem: the optimal strategy ${}^{\alpha}\widehat{X}$ is a dynamic version of the so-called fractional Kelly's strategy. It invests a fraction of wealth, which depends on the current level of drawdown, in the fund represented by $\widehat{X}$ and the remaining fraction in the baseline asset. When the domestic savings account is taken as the baseline asset, $\widehat{X}$ and ${}^{\alpha}\widehat{X}$ have the same instantaneous Sharpe ratio. Both portfolios $\widehat{X}$ and ${}^{\alpha}\widehat{X}$ are located at the Markowitz efficient frontier. However, ${}^{\alpha}\widehat{X}$ trades off long-term growth for a pathwise capital guarantee in the form of a drawdown constraint; in contrast, the portfolio $\widehat{X}$, or solutions to expected utility maximization in general, cannot offer such capital guarantee. Second, the optimal strategy ${}^{\alpha}\widehat{X}$ is given as a pathwise and model-independent transformation of the unconstrained numéraire strategy $\widehat{X}$. As a result, the optimal strategy disentangles the effects of model specification and risk attitude specification. The former yields the Kelly's strategy $\widehat{X}$. The latter specifies the

[1] Here, in the sense of *a* numéraire, e.g., currency.

transformation which is applied to $\widehat{X}$ to control the risk by avoiding drawdowns beyond a certain magnitude.

Detailed structural asymptotic properties of the optimal strategies ${}^{\alpha}\widehat{X}$ are also discussed; for example, we show that ${}^{\alpha}\widehat{X}$ is the only wealth process which may enjoy the numéraire property along increasing sequences of stopping times that tend to infinity. More importantly, a version of the so-called turnpike theorem is established: portfolios enjoying the numéraire property for investment with long time-horizons are close (in a strong sense) to ${}^{\alpha}\widehat{X}$ at initial times.

We stress the fact that the results presented here do not follow from previous literature because of the generality of our setup and the complex nature of drawdown constraints. In fact, novel characteristics appear in this setting. As was already mentioned, portfolios with the numéraire property, which maximize logarithmic utility, may depend on the financial planning horizon. Interestingly, the asymptotic solution does not depend on the way the financial planning horizon approaches infinity and is described explicitly, which is important from an investor's viewpoint. Furthermore, we emphasize that the findings of this paper are essentially model-independent and, therefore, rather robust. Finally, we wish to draw some attention to the underlying philosophy relative to the practical perspective. A long-run investor will only witness a single realization of the market dynamics. Therefore, pathwise outperformance is a very natural and appealing decision criterion. We show that it is possible to combine it with risk mitigation, which is done by restricting the universe of acceptable trajectories of wealth evolution and not by complicating the investor's decision criteria. This, we find, adds an interesting point to the debate in economics around Kelly's criterion, which is revisited later on in the text.

1.3. Mathematical Tools

To establish existence of the numéraire portfolio for an arbitrary time-horizon, we are inspired by existing results. We analyze the set of possible wealth outcomes at a specific time corresponding to wealth processes which satisfy the drawdown constraint and combine the classical optional decomposition theorem of Föllmer and Kramkov (1997) and Stricker and Yan (1998) with arguments from Kardaras (2010b) and Delbaen and Schachermayer (1994). However, classical arguments to show uniqueness fail since, in general, if we are given two strategies which satisfy the drawdown constraint and we follow one strategy up to a (stopping) time and then switch to the other we may violate the drawdown constraint. New arguments are developed which involve switching strategies at times when new maxima are attained.

An important tool throughout our study is the Azéma–Yor transformation, a result in stochastic analysis which allows one to build an explicit, model-independent, bijection between all wealth processes and wealth processes satisfying a drawdown constraint. This transformation was established in a general semimartingale setup in Carraro, El Karoui, and Oblój (2012) and used by Cherny and Obłój (2013) in a utility maximization setting. However, we note that a special case of it was already used in Cvitanić and Karatzas (1994). We show here that the Azéma–Yor transform ${}^{\alpha}\widehat{X}$ of $\widehat{X}$ has the numéraire property within the class of portfolios satisfying the α-drawdown constraint, both in an asymptotic sense and when sampled at times of its maximum. Since optimal strategies may depend on the time-horizon, it is not true that all other drawdown-constrained wealth processes are supermartingales in units on ${}^{\alpha}\widehat{X}$, a feature often used previously to define the numéraire property—see Long (1990) or Platen and Heath (2006).

1.4. Related Literature

As outlined previously, drawdown constraints have features appealing to various participants in financial markets and are often encountered in practice, in either explicit or implicit manner. Drawdown levels often serve as basis for performance measurement and are of prime importance both for investors and fund managers, see Browne and Kosowski (2010). A large drawdown could lead to a flight of capital from the fund, a threatening situation from a managerial perspective. Drawdown constraints may also result implicitly from the structure of hedge fund managers' incentives through the high-water mark provision—see, e.g., Guasoni and Obłój (2016).

Despite their practical importance, there are relatively few theoretical studies on portfolio selection with drawdown constraints. The main obstacle is the inherent difficulty associated with pathwise constraint which involves the running maximum process. Drawdown constraints were first considered in a continuous-time framework by Grossman and Zhou (1993), then by Cvitanić and Karatzas (1994) and more recently by Cherny and Obłój (2013). These contributions focused on maximizing the growth rate of expected utility and show that imposing drawdown constraints is essentially equivalent to changing investors' risk aversion. More precisely, Cherny and Obłój (2013) consider two investors in a general semimartingale model: one endowed with a power utility with risk aversion γ and facing an α-drawdown constraint and another with risk aversion $(\gamma + \alpha(1 - \gamma))$ and no constraints. They prove that the two are equivalent in the sense that they both achieve the same asymptotic growth rate of expected utility, and that their optimal portfolios are related through an explicit model-independent transformation.

Magdon-Ismail and Atiya (2004) derived results linking the maximum drawdown to average returns. In Chekhlov, Uryasev, and Zabarankin (2005), the problem of maximizing expected return subject to a risk constraint expressed in terms of the drawdown was considered and solved numerically in a simple discrete time setting. Finally, in continuous-time models, drawdown constraints were also recently incorporated into problems of maximizing expected utility from consumption—see Elie (2008) and Elie and Touzi (2008). Options on drawdowns were also explored as instruments to hedge against portfolio losses, see Vecer (2006). Furthermore, the maximization of growth subject to constraints arising from alternative risk measures is discussed in Pirvu and Žitković (2009).

While drawdown constraints are well motivated by market practice of assessing and reporting investment performance, the implication of the above works is that the parameter $\alpha \in [0, 1)$ is akin to risk aversion and allows for a flexible specification of attitude toward risk. As such, our results contribute to the debate whether an investor with a long financial planning horizon should use the growth-optimal strategy, as postulated by Kelly (1956). In this time-honored dispute (see MacLean, Thorp, and Ziemba 2011), the opposite sides were assumed by two among the most prominent scholars in the field: while Paul Samuelson fiercely criticized the use of Kelly's strategy, including the famous refute by Samuelson (1979) in words of one-syllable, Harry Markowitz argued for it already in his 1959 book (see also Markowitz 2006). The arguments in favor of Kelly's investment strategy rely on the fact that asymptotic growth should be of prime interest for long-run investment. The arguments against it point to the fact that the growth-rate maximization does not take into account investor's risk appetite and is too simplistic. Samuelson, as well as many others including seminal works of Merton (1971), looked instead at maximizing expected utility. While Kelly's strategy itself falls into this category, with the utility function being the logarithmic one, choices of other utility functions result in criteria that can accommodate different risk preference profiles. Our work may be interpreted as a way to merge the opposing sides: we adopt the pathwise outperformance as a very natural and appealing decision criterion but we show that it is possible to combine it with risk mitigation. However, the latter is done by restricting the universe of acceptable trajectories of wealth evolution, as opposed to elaborating on the investor's decision criteria.

1.5. Structure of the Paper

Section 2 contains a description of the financial market and introduces drawdown-constrained investments. In Section 3, the numéraire property of drawdown-constrained investments is explored. Main results are Theorem 3.5, establishing existence and uniqueness of portfolios with the numéraire property for finite time-horizons, and Theorem 3.8, which explicitly describes an investment that has the numéraire property at (stopping) times where it achieves its maximum—in particular, this includes its asymptotic numéraire property. More asymptotic optimality properties of the aforementioned investment are explored in Section 4: its asymptotic (or long-run) growth optimality is taken up in Theorem 4.1, and a strong result in the spirit of turnpike theorems is given in Theorem 4.7. Certain technical proofs are collected in Appendix A. Finally, in Appendix B we present an example in order to shed more light on the conclusion of the turnpike-type Theorem 4.7.

2. MARKET AND DRAWDOWN CONSTRAINTS

2.1. Financial Market

We consider a general frictionless financial market model with the only assumption of *continuous* price processes. Specifically, on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, where $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is a filtration satisfying the usual hypotheses of right-continuity and saturation by $\mathbb{P}$-null sets of $\mathcal{F}$, let $S = (S^1, \dots, S^d)$ be a d-dimensional semimartingale with a.s. continuous paths—see, for example, Karatzas and Shreve (1991). Each S^i, $i \in \{1, \dots, d\}$, is modeling the random movement of an asset price in the market. All the prices S^i are given in units of a fixed traded baseline asset. It is customary to assume that the baseline asset is the (domestic) savings account and then S^i are referred to as discounted prices, but in our context it is not important what units are fixed (i.e., which asset is taken as the baseline).

Define $\mathcal{X}$ to be the class of all *nonnegative* processes X of the form

$$X = 1 + \int_0^{\cdot} (H_t, \mathrm{d}S_t) = 1 + \int_0^{\cdot} \left(\sum_{i=1}^{d} H_t^i \mathrm{d}S_t^i \right), \tag{2.1}$$

where $H = (H^1, \dots, H^d)$ is a d-dimensional predictable and S-integrable[2] process. Throughout the paper, $(\cdot, \cdot)$ is used to (sometimes, formally) denote the inner product in $\mathbb{R}^d$. The process X of (2.1) represents the outcome of trading according to the

[2]All integrals are understood in the sense of vector stochastic integration. For this reason, we use notation such as $\int_0^{\cdot} (\sum_{i=1}^d H_t^i \mathrm{d}S_t^i)$ instead of $\sum_{i=1}^d \int_0^{\cdot} H_t^i \mathrm{d}S_t^i$, the latter corresponding to componentwise stochastic integration, which would only make sense if all stochastic integrals $\int_0^{\cdot} H_t^i \mathrm{d}S_t^i$ were well-defined for $i \in \{1, \dots, d\}$. Vector stochastic integration is more general and flexible than componentwise stochastic integration; its use has proved essential in order to formulate elegant versions of the Fundamental Theorem of Asset Pricing (Delbaen and Schachermayer 1994), as well as to ensure that optimal wealth processes exist—for example, it is crucial for the validity of Theorem A.1, which is used extensively throughout the paper.

investment strategy H, denominated in units of the baseline asset. In the sequel, we are interested in ratios of portfolios; therefore, the initial value X_0 plays no role as long as it is the same for all investment strategies. For convenience, we assume $X_0 = 1$ holds for all $X \in \mathcal{X}$.

In the following, we characterize in a precise manner the rich world of models that we permit for our market. These include most continuous-path models that have been studied in the literature. Essential is the existence of the (unconstrained) *numéraire portfolio*—see Long (1990). However, existence of an equivalent risk-neutral probability measure is not requested; therefore, certain forms of classical arbitrage are permitted.

DEFINITION 2.1. We shall say that there are opportunities for arbitrage of the first kind if there exist $T \in \mathbb{R}_+$ and an $\mathcal{F}_T$-measurable random variable ξ such that:

- $\mathbb{P}[\xi \geq 0] = 1$ and $\mathbb{P}[\xi > 0] > 0$;
- for all $x > 0$ there exists $X \in \mathcal{X}$, which may depend on x, with $\mathbb{P}[xX_T \geq \xi] = 1$.

The following mild and natural assumption is key to the development of the paper.

ASSUMPTION 2.2. *In the market described above, the following hold:*

(A1) There is no opportunity for arbitrage of the first kind.
(A2) There exists $X \in \mathcal{X}$ such that $\mathbb{P}[\lim_{t\to\infty} X_t = \infty] = 1$.

Condition (A1) in Assumption 2.2 is a minimal market viability assumption. On the other hand, condition (A2) asks for sufficient market growth in the long run. They are equivalent to the existence and growth condition of the numéraire portfolio.

THEOREM 2.3. *Condition (A1) of Assumption 2.2 is equivalent to:*

(B1) There exists $\widehat{X} \in \mathcal{X}$ such that $X/\widehat{X}$ is a (nonnegative) local martingale for all $X \in \mathcal{X}$.

Under the validity of (A1) or (B1), condition (A2) of Assumption 2.2 is equivalent to:

(B2) $\mathbb{P}[\lim_{t\to\infty} \widehat{X}_t = \infty] = 1$.

REMARK 2.4. The equivalence of (A1) and (B1) was first discussed in Long (1990). If the process $\widehat{X}$ in (B1) exists, then it is unique and is said to have the numéraire property. It is well known that it solves the log-utility maximization problem on any finite time-horizon, and that it achieves optimal asymptotic (or long-term) growth. We shall revisit these properties in a more general setting—see Remark 3.6 and Theorem 4.1.

The proof of Theorem 2.3 is given in Subsection A.1 of Appendix A. In fact, it is a special case of a more general Theorem A.1 therein which contains several useful equivalent conditions to the ones presented in Assumption 2.2.

2.2. Drawdown Constraints

To each wealth process $X \in \mathcal{X}$, we associate its *running maximum* process X^* defined via $X^*_t := \sup_{u\in[0,t]} X_u$ for $t \in \mathbb{R}_+$. The difference $X^* - X$ between the running maximum and the current wealth is called the *drawdown process*. As we argued in the introduction, different participants in financial markets may be interested to restrict the universe of their strategies to the ones which do not permit for drawdowns beyond a fixed fraction of the wealth's running maximum.

For any $\alpha \in [0, 1)$, we write ${}^{\alpha}\mathcal{X}$ for the class of wealth processes $X \in \mathcal{X}$ such that $X_t^* - X_t \leq (1-\alpha)X_t^*$, for all $t \geq 0$. Equivalently, $X \in {}^{\alpha}\mathcal{X}$ if and only if $X/X^* \geq \alpha$ holds identically. The $[0, 1]$-valued process X/X^* is called the *relative drawdown* process associated to X. It is clear that ${}^{\beta}\mathcal{X} \subseteq {}^{\alpha}\mathcal{X}$ for $0 \leq \alpha \leq \beta < 1$, and that ${}^{0}\mathcal{X} = \mathcal{X}$. Note that if $X \in \mathcal{X}$ satisfies $X \geq \alpha X^*$ on the interval $[0, T]$ (here, T can be any stopping time), then $(X_{T\wedge t})_{t\in\mathbb{R}_+} \in {}^{\alpha}\mathcal{X}$; therefore, it is appropriate to use ${}^{\alpha}\mathcal{X}$ as the set of wealth processes regardless of the investment horizon.

Interestingly, there is a one-to-one correspondence between wealth processes in $\mathcal{X}$ and wealth processes in ${}^{\alpha}\mathcal{X}$ for any $\alpha \in [0, 1)$. The bijection was derived explicitly in terms of the so-called Azéma–Yor processes in Carraro et al. (2012, theorem 3.4), and recently exploited in Cherny and Obłój (2013), in a general setting of possibly nonlinear drawdown constraints. This elegant machinery simplifies greatly in the case of "linear" drawdown constraints considered here, and we provide explicit arguments, similarly to the pioneering work of Cvitanić and Karatzas (1994). We first discuss how processes in $\mathcal{X}$ generate processes in ${}^{\alpha}\mathcal{X}$—the converse will be established in the proof of Proposition 2.5. For $X \in \mathcal{X}$ and $\alpha \in [0, 1)$, define a process ${}^{\alpha}X$ via

$$
{}^{\alpha}X := \alpha(X^*)^{1-\alpha} + (1-\alpha)X(X^*)^{-\alpha}. \tag{2.2}
$$

Using the fact that $\int_0^\infty \mathbb{I}_{\{X_t < X_t^*\}} \mathrm{d}X_t^* = 0$ a.s. holds, an application of Itô's formula gives

$$
{}^{\alpha}X = 1 + \int_0^{\cdot} (1-\alpha)(X_t^*)^{-\alpha} \mathrm{d}X_t, \tag{2.3}
$$

which implies that ${}^{\alpha}X \in \mathcal{X}$. Furthermore, (2.2) gives $\alpha(X^*)^{1-\alpha} \leq {}^{\alpha}X \leq (X^*)^{1-\alpha}$. Note also that times of maximum of X coincide with times of maximum of ${}^{\alpha}X$ and consequently ${}^{\alpha}X^* = (X^*)^{1-\alpha}$. It follows that

$$
\frac{{}^{\alpha}X}{{}^{\alpha}X^*} = \frac{\alpha(X^*)^{1-\alpha} + (1-\alpha)X(X^*)^{-\alpha}}{(X^*)^{1-\alpha}} = \alpha + (1-\alpha)\frac{X}{X^*} \geq \alpha, \tag{2.4}
$$

implying ${}^{\alpha}X \in {}^{\alpha}\mathcal{X}$. The converse is given by

PROPOSITION 2.5 (proposition 2.2 of Carraro et al. 2012). It holds that ${}^{\alpha}\mathcal{X} = \{{}^{\alpha}X \mid X \in \mathcal{X}\}$.

Proof. In the notation of Carraro et al. (2012), we have ${}^{\alpha}X = M^{F_\alpha}(X)$ with $F_\alpha : \mathbb{R}_+ \mapsto \mathbb{R}_+$ defined via $F_\alpha(x) = x^{1-\alpha}$ for $x \in \mathbb{R}_+$ and $X = M^{G_\alpha}({}^{\alpha}X)$ with $G_\alpha = F_\alpha^{-1}$. □

One can rewrite equation (2.3) in differential terms as

$$
\frac{\mathrm{d}{}^{\alpha}X_t}{{}^{\alpha}X_t} = \left(\frac{(1-\alpha)(X_t^*)^{-\alpha}X_t}{{}^{\alpha}X_t}\right)\frac{\mathrm{d}X_t}{X_t} = \frac{{}^{\alpha}X_t - \alpha\left({}^{\alpha}X_t^*\right)}{{}^{\alpha}X_t}\frac{\mathrm{d}X_t}{X_t}
$$

for $t < \inf\{u \in \mathbb{R}_+ \mid X_u = 0\} = \inf\{u \in \mathbb{R}_+ \mid {}^{\alpha}X_u - \alpha({}^{\alpha}X_u^*) = 0\}$. The above equation carries an important message: for $X \in \mathcal{X}$, the way that ${}^{\alpha}X$ is built is via investing a proportion

$$
{}^{\alpha}\pi^X := \frac{{}^{\alpha}X - \alpha\left({}^{\alpha}X^*\right)}{{}^{\alpha}X} = 1 - \frac{\alpha}{{}^{\alpha}X/{}^{\alpha}X^*} = \frac{(1-\alpha)(X/X^*)}{\alpha + (1-\alpha)(X/X^*)}
$$

in the fund represented by X, and the remaining proportion $1 - {}^{\alpha}\pi^X$ in the baseline asset. In particular, when the baseline asset is the domestic savings account, it follows that the Sharpe ratios of X and ${}^{\alpha}X$ are the same. Note that $0 \leq {}^{\alpha}\pi^X \leq 1-\alpha$ (so that $\alpha \leq 1 - {}^{\alpha}\pi^X \leq 1$). Furthermore, ${}^{\alpha}\pi^X$ depends only on $\alpha \in [0, 1)$ and the relative drawdown

X/X^* of X. In fact, the proportion ${}^{\alpha}\pi^X$ invested in the underlying fund represented by X is an increasing function of the relative drawdown X/X^*.

Recall the numéraire portfolio process $\widehat{X}$ in (B1) in Theorem A.1. When the above discussion is applied to ${}^{\alpha}\widehat{X}$, defined from $\widehat{X}$ via (2.2), it follows from Platen and Heath (2006, theorem 11.1.3 and corollary 11.1.4) that ${}^{\alpha}\widehat{X}$ is a locally optimal portfolio, in the sense that it locally maximizes the excess return over all investments with the same volatility. In view of (A.2), the wealth process $\widehat{X}$ is given explicitly in terms of the drift and quadratic covariation process of the multidimensional asset-price process. It follows that ${}^{\alpha}\widehat{X}$ for $\alpha \in [0, 1)$ is explicitly specified as well.

Even though the numéraire portfolio $\widehat{X}$ has optimal growth in an asymptotic sense (in this respect, see also Theorem 4.1), it is a quite risky investment. In fact, it experiences arbitrarily large flights of capital, as its relative drawdown process $\widehat{X}/\widehat{X}^*$ will become arbitrarily close to zero infinitely often. This is in fact equivalent to the following, seemingly more general statement, showing an oscillatory behavior of the relative drawdown for all wealth processes ${}^{\alpha}\widehat{X}$, $\alpha \in [0, 1)$.

PROPOSITION 2.6. *Under Assumption 2.2, it holds that*

$$\alpha = \liminf_{t\to\infty} \left(\frac{{}^{\alpha}\widehat{X}_t}{{}^{\alpha}\widehat{X}^*_t} \right) < \limsup_{t\to\infty} \left(\frac{{}^{\alpha}\widehat{X}_t}{{}^{\alpha}\widehat{X}^*_t} \right) = 1, \ \textit{a.s.} \quad \forall \alpha \in [0, 1).$$

The proof of Proposition 2.6 is given in Subsection A.2 of Appendix A.

3. THE NUMÉRAIRE PROPERTY

3.1. Expected Relative Return

Fix a stopping time T and $X, X' \in \mathcal{X}$, and define the *return of X relative to X' over the period* $[0, T]$ via

$$\mathrm{rr}_T(X|X') := \limsup_{t\to\infty} \left(\frac{X_{T\wedge t} - X'_{T\wedge t}}{X'_{T\wedge t}} \right) = \limsup_{t\to\infty} \left(\frac{X_{T\wedge t}}{X'_{T\wedge t}} \right) - 1.$$

(The convention $0/0 = 1$ is used throughout.) In other words, $\mathrm{rr}_T(X|X') = (X_T - X'_T)/X'_T$ holds on the event $\{T < \infty\}$, while $\mathrm{rr}_T(X|X') = \limsup_{t\to\infty}((X_t - X'_t)/X'_t) = \mathrm{rr}_\infty(\mathrm{rr}_T(X|X')$ holds on the event $\{T = \infty\}$. The above definition conveniently covers both cases. Observe that $\mathrm{rr}_T(X|X')$ is a $[-1, \infty]$-valued random variable. Therefore, for any stopping time T and $X, X' \in \mathcal{X}$, the quantity

$$\mathbb{E}\mathrm{rr}_T(X|X') := \mathbb{E}\left[\mathrm{rr}_T(X|X')\right]$$

is well defined and $[-1, \infty]$-valued. $\mathbb{E}\mathrm{rr}_T(X|X')$ represents the *expected return of X relative to X' over the time period* $[0, T]$.

The concept of expected relative returns is introduced for purposes of portfolio selection. A first idea that comes to mind is to proclaim that $X' \in \mathcal{X}$ is "strictly better" than $X \in \mathcal{X}$ for investment over the period $[0, T]$ if $\mathbb{E}\mathrm{rr}_T(X'|X) > 0$. However, this is not an appropriate notion: it is easy to construct examples where both $\mathbb{E}\mathrm{rr}_T(X'|X) > 0$ and $\mathbb{E}\mathrm{rr}_T(X|X') > 0$ hold. This fact is connected to *Siegel's paradox*—see Siegel (1972); more information is given in Remark 3.4. The reason is that, in general, $rr_T(X|X') \neq -X'|X)$. In fact, Proposition 3.3 implies that $\mathrm{rr}_T(X|X') \geq -X'|X)$, with equality holding only on the event $\{\lim_{t\to\infty}(X_{T\wedge t}/X'_{T\wedge t}) = 1\}$. A more appropriate definition would call $X' \in \mathcal{X}$

"strictly better" than $X \in \mathcal{X}$ for investment over the period $[0, T]$ if *both* $\mathbb{E}\mathrm{rr}_T(X'|X) > 0$ and $\mathbb{E}\mathrm{rr}_T(X|X') \leq 0$ hold. In fact, because of the inequality $\mathrm{rr}_T(X|X') \geq -\mathrm{rr}_T(X'|X)$, $\mathbb{E}\mathrm{rr}_T(X|X') \leq 0$ is enough to imply $\mathbb{E}\mathrm{rr}_T(X'|X) \geq 0$, and one has $\mathbb{E}\mathrm{rr}_T(X'|X) > 0$ in the case where $\mathbb{P}[\lim_{t\to\infty}(X_{T\wedge t}/X'_{T\wedge t}) = 1] < 1$.

The discussion of the previous paragraph can be summarized as follows: while positive expected returns of $X \in \mathcal{X}$ with respect to $X' \in \mathcal{X}$ do not imply that X is a better investment than X', we may regard nonpositive expected returns of $X \in \mathcal{X}$ with respect to $X' \in \mathcal{X}$ to indicate that X' is a better investment than X. Given the use of "lim sup" in the equality $\mathrm{rr}_T(X|X') = \limsup_{t\to\infty}((X_t - X'_t)/X'_t)$, valid on $\{T = \infty\}$, it seems particularly justified to regard X' as better than X when $\mathbb{E}\mathrm{rr}_\infty(X|X') \leq 0$ holds, at least in an asymptotic sense. We are led to the following concept:

DEFINITION 3.1. We say that X' has the numéraire property in a certain class of wealth processes for investment over the period $[0, T]$ if $\mathbb{E}\mathrm{rr}_T(X|X') \leq 0$ holds for all other X in the same class.

REMARK 3.2. The above definition is close in spirit to the numéraire in Long (1990). However following closely Long (1990) and the results pertaining to the nonconstrained case, one may be tempted to define the numéraire portfolio in a certain class of wealth processes by postulating that all other wealth processes in this class are supermartingales in units of the numéraire. However, in the context of drawdown constraints this would be a void concept as portfolios with the numéraire property may depend on the planning horizon—see Proposition 3.13.

The next result contains some useful properties of (expected) relative returns. In particular, it implies that the terminal value of an investment with the numéraire property within a certain class of processes for investment over a specified period of time is essentially unique.

PROPOSITION 3.3. *For any stopping time T, any $X \in \mathcal{X}$ and any $X' \in \mathcal{X}$, it holds that*

$$\mathrm{rr}_T(X'|X) \geq -\frac{\mathrm{rr}_T(X|X')}{1 + \mathrm{rr}_T(X|X')} \geq -\mathrm{rr}_T(X|X'),$$

with equality on $\{T < \infty\}$. Furthermore, the following equivalence is valid:

$$\mathbb{E}\mathrm{rr}_T(X'|X) \leq 0 \text{ and } \mathbb{E}\mathrm{rr}_T(X|X') \leq 0 \iff \mathbb{P}\left[\lim_{t\to\infty}\left(\frac{X_{T\wedge t}}{X'_{T\wedge t}}\right) = 1\right] = 1.$$

Proof. To begin with, note that

$$1 + \mathrm{rr}_T(X|X') = \limsup_{t\to\infty}\left(\frac{X_{T\wedge t}}{X'_{T\wedge t}}\right) \geq \left(\limsup_{t\to\infty}\left(\frac{X'_{T\wedge t}}{X_{T\wedge t}}\right)\right)^{-1} = \frac{1}{1 + \mathrm{rr}_T(X'|X)},$$

with equality holding on $\{T < \infty\}$. Continuing, we obtain

$$X|\mathrm{rr}_T(X') + \mathrm{rr}_T(X'|X) \geq \frac{1}{1 + \mathrm{rr}_T(X'|X)} - 1 + \mathrm{rr}_T(X'|X) = \frac{\mathrm{rr}_T(X'|X)^2}{1 + \mathrm{rr}_T(X'|X)}.$$

Upon interchanging the roles of X and X', we also obtain the corresponding inequality $\mathrm{rr}_T(X|X') + \mathrm{rr}_T(X'|X) \geq \mathrm{rr}_T(X|X')^2/(1 + \mathrm{rr}_T(X|X'))$; therefore,

$$\mathrm{rr}_T(X|X') + \mathrm{rr}_T(X'|X) \geq \frac{\mathrm{rr}_T(X'|X)^2}{1 + \mathrm{rr}_T(X'|X)} \vee \frac{\mathrm{rr}_T(X|X')^2}{1 + \mathrm{rr}_T(X|X')}. \tag{3.1}$$

It immediately follows that $\mathrm{rr}_T(X'|X) + \mathrm{rr}_T(X|X') \geq 0$. Therefore, by (3.1), the conditions $\mathbb{E}\mathrm{rr}_T(X'|X) \leq 0$ and $\mathbb{E}\mathrm{rr}_T(X|X') \leq 0$ are equivalent to $\mathbb{P}[\mathrm{rr}_T(X|X') = 0 = \mathrm{rr}_T(X'|X)] = 1$, which is in turn equivalent to $\mathbb{P}[\lim_{t\to\infty}(X_{T\wedge t}/X'_{T\wedge t}) = 1] = 1$. □

REMARK 3.4. The inequality $\mathbb{E}\mathrm{rr}_T(X|X') + \mathbb{E}\mathrm{rr}_T(X'|X) \geq 0$ for any stopping time T, any $X \in \mathcal{X}$ and any $X' \in \mathcal{X}$, appearing in the proof of Proposition 3.3 is known in the literature as Siegel's paradox. If X/X' has the interpretation of a currency exchange rate, Siegel's paradox states the following: given that expected relative rates of return are used as the objective, it is possible that a domestic investor may want to hold foreign currency for its positive expected return (in domestic units), while at the same a foreign investor wants to hold the domestic currency for its positive expected return (in foreign units). Equation (2.1) can be interpreted as a pointwise quantitative version of this paradox in terms of investment opportunities. The paragraph right before Definition 3.1 paves a way that avoids the pitfalls created by Siegel's paradox.

It follows from Proposition 3.3 that if $\mathbb{E}\mathrm{rr}_T(X|X') \leq 0$ and $\mathbb{E}\mathrm{rr}_T(X'|X) \leq 0$ both hold, then $X_T = X'_T$ a.s. on $\{T < \infty\}$, while $\lim_{t\to\infty}(X_t/X'_t) = 1$ a.s. on $\{T = \infty\}$, the latter being a version of "asymptotic equivalence" between X and X'.

The next result establishes existence of a process with the numéraire property in the class ${}^\alpha\mathcal{X}$ sampled at T for all $\alpha \in [0, 1)$ and finite time-horizon T, and shows that such process is uniquely defined on the stochastic interval $[0, T] = \{(\omega, t) \in \Omega \times \mathbb{R}_+ \mid 0 \leq t \leq T(\omega)\}$. (Note that the latter uniqueness property is stronger than plain uniqueness of the terminal value of processes with the numéraire property that is guaranteed by Proposition 3.3.) Theorem 3.8 will address the possibility of an infinite time-horizon.

THEOREM 3.5. *Let T be a stopping time with* $\mathbb{P}[T < \infty] = 1$. *Under condition (A1) of Assumption 2.2, there exists $\widetilde{X} \in {}^\alpha\mathcal{X}$, which may depend on T, such that* $\mathbb{E}\mathrm{rr}_T(X|\widetilde{X}) \leq 0$ *holds for all $X \in {}^\alpha\mathcal{X}$. Furthermore, $\widetilde{X}$ has the following uniqueness property: for any other process $\widetilde{Z} \in {}^\alpha\mathcal{X}$ such that* $\mathbb{E}\mathrm{rr}_T(X|\widetilde{Z}) \leq 0$ *holds for all $X \in {}^\alpha\mathcal{X}$, $\widetilde{X} = \widetilde{Z}$ a.s. holds on* $[0, T]$.

The proof of Theorem 3.5 is given in Subsection A.3 of Appendix A.

REMARK 3.6. In the notation of Theorem 3.5, the log-utility maximization problem at time T is solved by the wealth process ${}^\alpha\widetilde{X}$. Indeed, the inequality $\log(x) \leq x - 1$, valid for all $x \in \mathbb{R}_+$, gives

$$\mathbb{E}\left[\log\left(\frac{X_T}{\widetilde{X}_T}\right)\right] \leq \mathbb{E}\left[\frac{X_T}{\widetilde{X}_T} - 1\right] = \mathbb{E}\mathrm{rr}_T(X|\widetilde{X}) \leq 0$$

for all $X \in {}^\alpha\mathcal{X}$. This is a version of relative expected log-optimality, which turns to actual expected log-optimality as soon as the expected log-maximization problem is well-posed—in this respect, see also Karatzas and Kardaras (2007, subsection 3.7).

In view of Theorem 3.5, the above discussion ensures existence and uniqueness of expected log-utility optimal wealth processes for finite time-horizons in a drawdown-constrained investment framework. To the best of the authors' knowledge, results regarding existence and uniqueness of optimal processes for utility maximization problems involving finite time-horizon and drawdown constraints are absent from the literature.

3.2. The Numéraire Property at Times of Maximum of the Numéraire Portfolio

When $\alpha = 0$, the fact that $X/\widehat{X}$ is a nonnegative supermartingale and the optional sampling theorem imply that $\mathbb{E}\mathrm{rr}_T(X|\widehat{X}) \leq 0$ holds for all stopping times T and all $X \in \mathcal{X}$.

Therefore, the process $\widehat{X}$ has a "global" (in time) numéraire property. Furthermore, the supermartingale convergence theorem implies that $\lim_{t\to\infty}(X_t/\widehat{X}_t)$ $\mathbb{P}$-a.s. exists for all $X \in \mathcal{X}$; therefore,

$$\mathrm{rr}_\infty(X|\widehat{X}) = \lim_{t\to\infty}\left(\frac{X_t - \widehat{X}_t}{\widehat{X}_t}\right) = \lim_{t\to\infty}\left(\frac{X_t}{\widehat{X}_t}\right) - 1. \tag{3.2}$$

For finite time-horizons, the situation is more complicated for $\alpha \in (0, 1)$. In Theorem 3.8, we shall see that ${}^\alpha\widehat{X}$ has the numéraire property in ${}^\alpha\mathcal{X}$ for certain stopping times (which include the asymptotic case $T = \infty$). However, ${}^\alpha\widehat{X}$ does not have the numéraire property for all finite time-horizons, as Proposition 3.13 shows.

We continue the development by defining a class of stopping times which will be important in the sequel.

DEFINITION 3.7. A stopping time τ will be called a *time of maximum* of $\widehat{X}$ if $\widehat{X}_\tau = \widehat{X}^*_\tau$ holds a.s. on the event $\{\tau < \infty\}$.

A couple of remarks are in order. First, from (2.2) one can immediately see that times of maximum of $\widehat{X}$ are also times of maximum of ${}^\alpha\widehat{X}$ for all $\alpha \in [0, 1)$. Second, the restriction in the definition of a time τ of maximum of $\widehat{X}$ is only enforced on $\{\tau < \infty\}$. Under Assumption 2.2, and in view of Theorem A.1, one has $\widehat{X}_\tau = \widehat{X}^*_\tau = \infty$ holding a.s. on $\{\tau = \infty\}$. For this reason, $\tau = \infty$ is an important special case of a time of maximum of $\widehat{X}$.

The following theorem, the second main result of this section, establishes the *numéraire property* of ${}^\alpha\widehat{X}$ in ${}^\alpha\mathcal{X}$ over $[0, \infty]$ or, more generally, over $[0, \tau]$ for any time τ of maximum of $\widehat{X}$. We recall that ${}^\alpha\mathcal{X} = \{{}^\alpha X \mid X \in \mathcal{X}\}$.

THEOREM 3.8. *Recall that ${}^\alpha\widehat{X} \in {}^\alpha\mathcal{X}$ is defined from $\widehat{X}$ via* (2.2). *Under Assumption 2.2, for any $\alpha \in [0, 1)$ and $X \in \mathcal{X}$, we have:*

(1) $\lim_{t\to\infty}({}^\alpha X_t/{}^\alpha\widehat{X}_t)$ *a.s. exists in $\mathbb{R}_+$. Moreover,*

$$\mathrm{rr}_\infty({}^\alpha X|{}^\alpha\widehat{X}) = \left(\lim_{t\to\infty}\left(\frac{X_t}{\widehat{X}_t}\right)\right)^{1-\alpha} - 1 = \left(1 + \mathrm{rr}_\infty\left(X|\widehat{X}\right)\right)^{1-\alpha} - 1. \tag{3.3}$$

(2) For σ and τ two times of maximum of $\widehat{X}$ with $\sigma \le \tau$, it a.s. holds that

$$\mathbb{E}\left[\mathrm{rr}_\tau\left({}^\alpha X|{}^\alpha\widehat{X}\right) \mid \mathcal{F}_\sigma\right] \le \mathrm{rr}_\sigma\left({}^\alpha X|{}^\alpha\widehat{X}\right). \tag{3.4}$$

In particular, letting $\sigma = 0$, $\mathbb{E}\mathrm{rr}_\tau(Z|{}^\alpha\widehat{X}) \le 0$ holds for any $\alpha \in [0, 1)$ and $Z \in {}^\alpha\mathcal{X}$.

We proceed with several remarks on the implications of Theorem 3.8, the proof of which is given in Subsection A.4 of Appendix A.

REMARK 3.9. The existence of the limit in (3.2) is guaranteed by the nonnegative supermartingale convergence theorem. In contrast, proving that $\lim_{t\to\infty}({}^\alpha X_t/{}^\alpha\widehat{X}_t)$ exists a.s. for $X \in \mathcal{X}$ and $\alpha \in (0, 1)$ is more involved, since, in general, the process ${}^\alpha X/{}^\alpha\widehat{X}$ does not have the supermartingale property. In fact, the existence of the latter limit is proved together with the asymptotic relationship (3.3). Note, however, that an analogue of the supermartingale property is provided by statement (2) of Theorem 3.8. Indeed, (3.4) implies that, when sampled at an increasing sequence of times of maximum of $\widehat{X}$, the process ${}^\alpha X/{}^\alpha\widehat{X}$ is a supermartingale along these times for all $X \in \mathcal{X}$.

REMARK 3.10. Given statement (1) of Theorem 3.8, the fact that $\mathbb{E}\mathrm{rr}_\infty({}^\alpha X|{}^\alpha\widehat{X}) \le 0$ holds for any $X \in \mathcal{X}$ and $\alpha \in [0, 1)$ is a simple consequence of Jensen's inequality. Indeed, for any $X \in \mathcal{X}$,

$$\begin{aligned}\mathbb{E}\mathrm{rr}_\infty\left({}^\alpha X|{}^\alpha\widehat{X}\right) = &\ \mathbb{E}\left[\left(1+\mathrm{rr}_\infty(X|\widehat{X})\right)^{1-\alpha}\right]-1\\ \le &\ \left(\mathbb{E}\left[1+\mathrm{rr}_\infty(X|\widehat{X})\right]\right)^{1-\alpha}-1\\ = &\ \left(1+\mathbb{E}\mathrm{rr}_\infty(X|\widehat{X})\right)^{1-\alpha}-1 \le 0.\end{aligned}$$

The full proof of statement (2) of Theorem 3.8, given in Appendix A, is more involved.

REMARK 3.11. The fact that $\mathrm{rr}_\infty({}^\alpha X|{}^\alpha\widehat{X}) = (1+\mathrm{rr}_\infty(X|\widehat{X}))^{1-\alpha}-1$ holds for all $\alpha \in [0, 1)$ can be easily seen to imply that $|\mathrm{rr}_\infty({}^\beta X|{}^\beta\widehat{X})| \le |\mathrm{rr}_\infty({}^\alpha X|{}^\alpha\widehat{X})|$ holds whenever $0 \le \alpha \le \beta < 1$. In other words, using the same generating wealth process X and enforcing harsher drawdown constraints reduces the (asymptotic) difference in the performance of the drawdown-constrained process ${}^\alpha X$ against the long-run optimum ${}^\alpha\widehat{X}$.

REMARK 3.12. Let us consider the hitting times of $\widehat{X}$, parameterized on the logarithmic scale:

$$\tau_\ell := \inf\{t \in \mathbb{R}_+ \mid \widehat{X}_t = \exp(\ell)\}, \quad \ell \in \mathbb{R}_+. \tag{3.5}$$

Note that τ_ℓ is a time of maximum of $\widehat{X}$. Since times of maximum of $\widehat{X}$ coincide with times of maximum of ${}^\alpha\widehat{X}$ for $\alpha \in [0, 1)$, $\tau_\ell = \inf\{t \in \mathbb{R}_+ \mid {}^\alpha\widehat{X}_t = \exp((1-\alpha)\ell)\}$ holds for all $\alpha \in [0, 1)$. According to Assumption 2.2, $\mathbb{P}[\tau_\ell < \infty] = 1$ holds for all $\ell \in \mathbb{R}_+$.

By Remark 3.6, the log-utility maximization problem at time τ_ℓ for the class ${}^\alpha\mathcal{X}$ is solved by the wealth process ${}^\alpha\widehat{X}$. Moreover, assume that $U : \mathbb{R}_+ \mapsto \mathbb{R} \cup \{-\infty\}$ is any increasing and concave function such that $U(x) > -\infty$ for all $x \in (0, \infty)$. Jensen's inequality implies that

$$\mathbb{E}\left[U\left({}^\alpha X_{\tau_\ell}\right)\right] \le U\left(\mathbb{E}\left[{}^\alpha X_{\tau_\ell}\right]\right) \le U(\exp((1-\alpha)\ell)) = \mathbb{E}\left[U\left({}^\alpha\widehat{X}_{\tau_\ell}\right)\right], \quad \text{for all } X \in \mathcal{X}.$$

It follows that any (and not only the logarithmic) utility maximization problem at a hitting time τ_ℓ for the class ${}^\alpha\mathcal{X}$ is solved by the wealth process ${}^\alpha\widehat{X}$. This is a remarkable fact that is extremely robust, since it does not require any modeling assumptions.

Theorem 3.8, coupled with some simple observations, implies that drawdown-constrained portfolios with the numéraire property may depend on the time-horizon. This fact, which was already hinted in several places (for example, in Remark 3.2), is established below.

PROPOSITION 3.13. *Under Assumption 2.2, there exist $X \in {}^\alpha\mathcal{X}$ and stopping times T and τ with $\mathbb{P}[T < \tau < \infty] = 1$ such that X has the numéraire property in ${}^\alpha\mathcal{X}$ over the investment period* $[0, \tau]$*, while X fails to have the numéraire property in ${}^\alpha\mathcal{X}$ over the investment period* $[0, T]$.

Proof. Fix $\alpha \in (0, 1)$. Define $T := \inf\{t \in (0, \infty) \mid \widehat{X}_t/\widehat{X}^*_t = \alpha\}$, and observe that Proposition 2.6 implies that $\mathbb{P}[T < \infty] = 1$ holds. Furthermore, define $\tau := \inf\{t \in (T, \infty) \mid \widehat{X}_t = \widehat{X}^*_T\}$. Clearly, τ is a time of maximum of $\widehat{X}$; furthermore, Assumption 2.2 implies that $\mathbb{P}[\tau < \infty] = 1$. In view of Theorem 3.8, ${}^\alpha\widehat{X} \in {}^\alpha\mathcal{X}$ has the numéraire property in ${}^\alpha\mathcal{X}$ over the investment period $[0, \tau]$.

Continuing, note that $\widehat{X}^T \equiv \widehat{X}_{T\wedge\cdot} \in {}^\alpha\mathcal{X}$. The numéraire property of $\widehat{X}$ in $\mathcal{X} \supseteq {}^\alpha\mathcal{X}$ implies that $\mathbb{E}\mathrm{rr}_T(X|\widehat{X}^T) \le 0$ holds for all $X \in {}^\alpha\mathcal{X}$, resulting in the numéraire property of

$\widehat{X}^T$ in ${}^\alpha\mathcal{X}$ over the investment period $[0, T]$. Since $\mathbb{P}[{}^\alpha\widehat{X}_T = \widehat{X}_T] = 0$, it follows that ${}^\alpha\widehat{X}^T$ fails to have the numéraire property in ${}^\alpha\mathcal{X}$ over the investment period $[0, T]$. □

With the stopping time T as defined in the proof of Proposition 3.13, note the following: if one follows the nonconstrained numéraire portfolio $\widehat{X}$ up to T, the drawdown constraints will mean that one has to invest all capital in the baseline account from time T onward. It is clear that this strategy will not be long-run optimal. Further discussion on the subject of long-run optimality is given in Section 4.

4. MORE ON ASYMPTOTIC OPTIMALITY

4.1. Maximization of Long-Term Growth

The next theorem is concerned with the asymptotic growth-optimality property of ${}^\alpha\widehat{X}$ in ${}^\alpha\mathcal{X}$ for $\alpha \in [0, 1)$. It extends the result of Cvitanić and Karatzas (1994, section 7) to a more general setting and with a simpler proof. In the subsequent subsection we continue with a considerably finer analysis relating the finite-time and asymptotic optimality of ${}^\alpha\widehat{X}$ in ${}^\alpha\mathcal{X}$.

One of the equivalent conditions to (A1) of Assumption 2.2 is that a market-growth process G exists: G is a nonnegative and nondecreasing process such that $\log(\widehat{X}) = G + L$ for a local martingale L; furthermore, Assumption (A2) is equivalent to $\lim_{t\to\infty} G_t = \infty$; see Theorem A.1. In models using Itô processes, $2G$ is the integrated squared risk-premium in the market, see Remark A.2. Our next result implies that one can use G to control the growth rate of any portfolio.

THEOREM 4.1. *Under Assumption 2.2, for any $Z \in {}^\alpha\mathcal{X}$ we a.s. have that*

$$\limsup_{t\to\infty}\left(\frac{1}{G_t}\log(Z_t)\right) \le 1-\alpha = \lim_{t\to\infty}\left(\frac{1}{G_t}\log\left({}^\alpha\widehat{X}_t\right)\right). \tag{4.1}$$

Proof. The fact that $\lim_{t\to\infty}(\log(\widehat{X}_t)/G_t) = 1$ holds on the event $\{G_\infty = \infty\}$ was established in the proof of Theorem A.1. Again, in view of Theorem A.1, condition (A2) of Assumption 2.2 is equivalent to $\mathbb{P}[G_\infty = \infty] = 1$; therefore, a.s.,

$$\lim_{t\to\infty}\left(\frac{1}{G_t}\log(\widehat{X}_t)\right) = 1.$$

Observe that by concavity of the function $\mathbb{R}_+ \ni x \mapsto x^{1-\alpha}$, ${}^\alpha\widehat{X} \ge \widehat{X}^{1-\alpha}$ holds. Combining this with the above yields, a.s.,

$$\liminf_{t\to\infty}\left(\frac{1}{G_t}\log\left({}^\alpha\widehat{X}_t\right)\right) \ge 1-\alpha.$$

On the other hand, since G is nondecreasing and ${}^\alpha\widehat{X}$ achieves maximum values at the times $(\tau_\ell)_{\ell\in\mathbb{R}_+}$ of (3.5), it holds a.s. that

$$\begin{aligned}
\limsup\nolimits_{t\to\infty}\left(\tfrac{1}{G_t}\log\left({}^\alpha\widehat{X}_t\right)\right) &= \limsup\nolimits_{\ell\to\infty}\left(\tfrac{1}{G_{\tau_\ell}}\log\left({}^\alpha\widehat{X}_{\tau_\ell}\right)\right)\\
&= (1-\alpha)\limsup\nolimits_{\ell\to\infty}\left(\tfrac{\ell}{G_{\tau_\ell}}\right)\\
&= (1-\alpha)\limsup\nolimits_{\ell\to\infty}\left(\tfrac{1}{G_{\tau_\ell}}\log\left(\widehat{X}_{\tau_\ell}\right)\right) = 1-\alpha.
\end{aligned}$$

It follows that, a.s.,

$$\lim_{t\to\infty}\left(\frac{1}{G_t}\log\left({}^{\alpha}\widehat{X}_t\right)\right) = 1-\alpha.$$

Fix $Z \in {}^{\alpha}\mathcal{X}$. The full result of Theorem 4.1 now follows immediately upon noticing that, a.s.,

$$\limsup_{t\to\infty}\left(\frac{1}{G_t}\log\left(\frac{Z_t}{{}^{\alpha}\widehat{X}_t}\right)\right) \leq 0,$$

which is valid in view of the facts that $\mathbb{P}[G_\infty = \infty] = 1$ and $\mathbb{P}[\mathrm{rr}_\infty(Z|{}^{\alpha}\widehat{X}) < \infty] = 1$, the latter following from the inequality $\mathbb{E}\mathrm{rr}_\infty(Z|{}^{\alpha}\widehat{X}) \leq 0$, which was established in Theorem 3.8. □

REMARK 4.2. Fix $\alpha \in [0, 1)$. In the setting of Theorem 4.1, any ${}^{\alpha}X \in {}^{\alpha}\mathcal{X}$ such that, a.s.,

$$\lim_{t\to\infty}\left(\frac{1}{G_t}\log\left(\frac{{}^{\alpha}X_t}{{}^{\alpha}\widehat{X}_t}\right)\right) = 0$$

also enjoys the asymptotic growth-optimality property in the sense of achieving equality in (4.1). As a simple example, let $X \in \mathcal{X}$, $\kappa \in (0, 1)$ and $\widetilde{X} := \kappa\widehat{X} + (1-\kappa)X$. Then $\widetilde{X}^* \geq \kappa\widehat{X}^*$ so that ${}^{\alpha}\widetilde{X} \geq \alpha{}^{\alpha}\widetilde{X}^* \geq \alpha(\kappa\widehat{X}^*)^{1-\alpha} = (\alpha\kappa^{1-\alpha}){}^{\alpha}\widehat{X}^* \geq (\alpha\kappa^{1-\alpha}){}^{\alpha}\widehat{X}$ and, consequently, ${}^{\alpha}\widetilde{X}$ enjoys the asymptotic growth optimality. In contrast, the asymptotic numéraire property is much stronger. Combining Theorem 3.8 and Proposition 3.3, it follows that if ${}^{\alpha}X \in {}^{\alpha}\mathcal{X}$ is to have the asymptotic numéraire property, then the much stronger "asymptotic equivalence" condition $\lim_{t\to\infty}({}^{\alpha}X_t/{}^{\alpha}\widehat{X}_t) = 1$ has to be a.s. valid. We shall see below that this in fact implies the even stronger condition ${}^{\alpha}X = {}^{\alpha}\widehat{X}$.

4.2. Optimality through Sequences of Stopping Times Converging to Infinity

By Theorem 3.8, $\mathbb{E}\mathrm{rr}_\infty({}^{\alpha}X|{}^{\alpha}\widehat{X}) \leq 0$ holds for all $X \in \mathcal{X}$, a result which can be interpreted as long-run numéraire optimality property of ${}^{\alpha}\widehat{X}$ in ${}^{\alpha}\mathcal{X}$. However, in effect, this result assumes that the investment time-horizon is actually equal to infinity. On both theoretical and practical levels, one may be rather interested in considering a sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ that converge to infinity and examine the behavior of optimal wealth processes (in the numéraire sense) in the limit. We present two results in this direction. Proposition 4.3 establishes that the only process in ${}^{\alpha}\mathcal{X}$ possessing the numéraire property along an increasing sequence of stopping times tending to infinity is ${}^{\alpha}\widehat{X}$. The second result, Theorem 4.7, is more delicate than Proposition 4.3, and may be regarded as a version of so-called *turnpike theorems*, an appellation coined in Leland (1972). While the traditional formulation of turnpike theorems involves two investors with long financial planning horizon and similar preferences for large levels of wealth, Theorem 4.7 compares a portfolio having the numéraire property for a long, but finite, time-horizon with the corresponding portfolio having the asymptotic numéraire property. Loosely speaking, Theorem 4.7 states that, when the time-horizon T is long, the process ${}^{\alpha}\widetilde{X}$ that has the numéraire property in ${}^{\alpha}\mathcal{X}$ for investment over the interval $[0, T]$ will be very close initially (in time) to ${}^{\alpha}\widehat{X}$ in a very strong sense.

PROPOSITION 4.3. *Under the validity of Assumption 2.2, suppose that there exist* $X \in \mathcal{X}$ *and a sequence of (possibly infinite-valued) stopping times* $(T_n)_{n\in\mathbb{N}}$ *with* $\lim_{n\to\infty}\mathbb{P}[T_n > t] = 1$ *holding for all* $t \in \mathbb{R}_+$, *such that* $\liminf_{n\to\infty}\mathbb{E}\mathrm{rr}_{T_n}({}^{\alpha}\widehat{X}|{}^{\alpha}X) \leq 0$. *Then,* ${}^{\alpha}X = {}^{\alpha}\widehat{X}$.

Proof. Upon passing to a subsequence of $(T_n)_{n\in\mathbb{N}}$ if necessary, we may assume without loss of generality that $\mathbb{P}[\lim_{n\to\infty} T_n = \infty] = 1$. Then, by Theorem 3.8, $\lim_{t\to\infty}({}^\alpha\widehat{X}_t/{}^\alpha X_t)$ exists a.s. in $(0, \infty]$ and a use of Fatou's lemma gives

$$\begin{aligned}\mathbb{E}\left[\lim_{t\to\infty}\left(\frac{{}^\alpha\widehat{X}_t}{{}^\alpha X_t}\right)\right] &= \mathbb{E}\left[\liminf_{n\to\infty}\left(\frac{{}^\alpha\widehat{X}_{T_n}}{{}^\alpha X_{T_n}}\right)\right] \le \liminf_{n\to\infty}\left(\mathbb{E}\left[\frac{{}^\alpha\widehat{X}_{T_n}}{{}^\alpha X_{T_n}}\right]\right)\\ &= 1 + \liminf_{n\to\infty}\mathbb{E}\mathrm{rr}_{T_n}({}^\alpha\widehat{X}|{}^\alpha X) \le 1.\end{aligned}$$

Since we have both $\mathbb{E}[\lim_{t\to\infty}({}^\alpha\widehat{X}_t/{}^\alpha X_t)] \le 1$ and $\mathbb{E}[\lim_{t\to\infty}({}^\alpha X_t/{}^\alpha\widehat{X}_t)] \le 1$ holding, Jensen's inequality implies that $\lim_{t\to\infty}({}^\alpha X_t/{}^\alpha\widehat{X}_t) = 1$ a.s. holds. By Theorem 3.8, $\lim_{t\to\infty}(X_t/\widehat{X}_t) = 1$ a.s. holds. This fact, combined with the conditional form of Fatou's lemma and the supermartingale property of $X/\widehat{X}$ gives $X_t/\widehat{X}_t \ge 1$ a.s. for each $t \in \mathbb{R}_+$. Combined with $\mathbb{E}[X_t/\widehat{X}_t] \le 1$, this gives $\widehat{X}_t = X_t$ a.s. for all $t \in \mathbb{R}_+$. The path-continuity of the process $X/\widehat{X}$ implies that $X = \widehat{X}$, i.e., that ${}^\alpha X = {}^\alpha\widehat{X}$. □

REMARK 4.4. Following the same reasoning as in the proof of Proposition 4.3, one can also show that if τ is a time of maximum of $\widehat{X}$ and $\mathbb{E}\mathrm{rr}_\tau({}^\alpha\widehat{X}|{}^\alpha X) \le 0$ holds for some $X \in \mathcal{X}$, then ${}^\alpha X = {}^\alpha\widehat{X}$ holds identically on the stochastic interval $[0, \tau]$.

In order to state Theorem 4.7, we define a strong notion of convergence in the space of semimartingales, introduced in Emery (1979).

DEFINITION 4.5. For a stopping time T, say that a sequence $(\xi^n)_{n\in\mathbb{N}}$ of semimartingales converges over $[0, T]$ in the Emery topology to another semimartingale ξ, and write $\mathcal{S}_T\text{-}\lim_{n\to\infty}\xi^n = \xi$, if

$$\lim_{n\to\infty}\sup_{\eta\in\mathcal{P}_1}\mathbb{P}\left[\sup_{t\in[0,T]}\left|\eta_0(\xi_0^n - \xi_0) + \int_0^t \eta_s \mathrm{d}\xi_s^n - \int_0^t \eta_s \mathrm{d}\xi_s\right| > \epsilon\right] = 0 \tag{4.2}$$

holds for all $\epsilon > 0$, where $\mathcal{P}_1$ denotes the set of all predictable processes η with $\sup_{t\in\mathbb{R}_+}|\eta_t| \le 1$. Furthermore, we say that the sequence $(\xi^n)_{n\in\mathbb{N}}$ of semimartingales converges locally in the Emery topology to another semimartingale ξ, and write $\mathcal{S}_{\mathrm{loc}}\text{-}\lim_{n\to\infty}\xi^n = \xi$, if $\mathcal{S}_T\text{-}\lim_{n\to\infty}\xi^n = \xi$ holds for all a.s. finitely valued stopping times T.

REMARK 4.6. In the setting of Definition 4.5, assume that $(\xi^n)_{n\in\mathbb{N}}$ converges locally in the Emery topology to ξ. By taking $\eta \equiv 1$ in (4.2), we see that

$$\lim_{n\to\infty}\mathbb{P}\left[\sup_{t\in[0,T]}|\xi_t^n - \xi_t| > \epsilon\right] = 0$$

holds for all $\epsilon > 0$ and all a.s. finitely valued stopping times T. In other words, the sequence $(\xi^n)_{n\in\mathbb{N}}$ converges in *probability, uniformly on compacts,* to ξ.

THEOREM 4.7. *Suppose that $(T_n)_{n\in\mathbb{N}}$ is a sequence of stopping times such that $\lim_{n\to\infty}\mathbb{P}[T_n > t] = 1$ holds for all $t \in \mathbb{R}_+$. For each $n \in \mathbb{N}$, let ${}^\alpha\widetilde{X}^n \in {}^\alpha\mathcal{X}$ have the numéraire property in ${}^\alpha\mathcal{X}$ for investment over the period* $[0, T_n]$. *Under Assumption 2.2, it holds that $\mathcal{S}_{\mathrm{loc}}\text{-}\lim_{n\to\infty}{}^\alpha\widetilde{X}^n = {}^\alpha\widehat{X}$.*

The proof of Theorem 4.7 is given in Subsection A.5 of Appendix A.

REMARK 4.8. In the setting of Theorem 4.7, the fact that $\mathcal{S}_{\mathrm{loc}}\text{-}\lim_{n\to\infty}{}^\alpha\widetilde{X}^n = {}^\alpha\widehat{X}$ implies by proposition 2.9 in Kardaras (2013) that $\lim_{n\to\infty}\mathbb{P}[[{}^\alpha\widetilde{X}^n - {}^\alpha\widehat{X}, {}^\alpha\widetilde{X}^n - {}^\alpha\widehat{X}]_T > \epsilon] = 0$

holds for all a.s. finitely valued stopping times T and $\epsilon > 0$. Writing ${}^{\alpha}\widehat{X} = 1 + \int_0^{\cdot} ({}^{\alpha}\widehat{H}_t, \mathrm{d}S_t)$ and ${}^{\alpha}\widetilde{X}^n = 1 + \int_0^{\cdot} ({}^{\alpha}\widetilde{H}_t^n, \mathrm{d}S_t)$ for all $n \in \mathbb{N}$ for appropriate d-dimensional strategies ${}^{\alpha}\widehat{H}$ and $({}^{\alpha}\widetilde{H}^n)_{n\in\mathbb{N}}$, we obtain

$$\lim_{n\to\infty} \mathbb{P}\left[\int_0^T \left({}^{\alpha}\widetilde{H}_t^n - {}^{\alpha}\widehat{H}_t,\ \mathrm{d}[S,S]_t \left({}^{\alpha}\widetilde{H}_t^n - {}^{\alpha}\widehat{H}_t\right)\right) > \epsilon\right] = 0$$

for all a.s. finitely valued stopping times T and $\epsilon > 0$. The previous relation implies that it is not only wealth that converges to the limiting one in each finite time-interval—the corresponding employed strategy does so as well.

REMARK 4.9. In the setting of Theorem 4.7, the conclusion is that convergence of ${}^{\alpha}\widetilde{X}^n$ to ${}^{\alpha}\widehat{X}$ holds over finite time-intervals that do not depend on $n \in \mathbb{N}$. One can ask whether the whole wealth process ${}^{\alpha}\widetilde{X}^n$ is close to ${}^{\alpha}\widehat{X}$ over the stochastic interval $[0, T_n]$ for each $n \in \mathbb{N}$. This is not true in general; in Appendix B we present an example, valid under all models for which Assumption 2.2 holds, where the ratio ${}^{\alpha}\widetilde{X}^n_{T_n} / {}^{\alpha}\widehat{X}_{T_n}$ as $n \to \infty$ oscillates between $1/(2-\alpha)$ and ∞. Note that the example only covers cases where $\alpha \in (0, 1)$; if $\alpha = 0$, ${}^{\alpha}\widetilde{X}^n = {}^{\alpha}\widehat{X}$ always holds for all $n \in \mathbb{N}$.

APPENDIX A: TECHNICAL PROOFS

We start by describing in Subsection A.1 several useful equivalent formulations of Assumption 2.2. Thereafter, through the course of Appendix A, the validity of Assumption 2.2 is always in force. The only exception is Subsection A.3, where only the condition (A1) of Assumption 2.2 is required.

A.1. Equivalent Conditions to Assumption 2.2.

Recall the market specification in Section 2.1. For $i \in \{1, \ldots, d\}$ write $S^i = S_0^i + B^i + M^i$ for the Doob–Meyer decomposition of S^i into a continuous finite variation process B^i with $B_0^i = 0$ and a local martingale M^i with $M_0^i = 0$. For $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, d\}$, $[S^i, S^j] = [M^i, M^j]$ denotes the covariation process of S^i and S^j.

The following result follows theorem 4 of Kardaras (2010a) and contains useful equivalent conditions to the ones presented in Assumption 2.2.

THEOREM A.1. *Condition (A1) of Assumption 2.2 is equivalent to any of the following:*

(B1) There exists $\widehat{X} \in \mathcal{X}$ such that $X/\widehat{X}$ is a (nonnegative) local martingale for all $X \in \mathcal{X}$.

(C1) There exists a d-dimensional process ρ such that $B^i = \int_0^{\cdot} \sum_{j=1}^d \rho_t^j \mathrm{d}[S^j, S^i]_t$ holds for each $i \in \{1, \ldots, d\}$. Furthermore, the nonnegative and nondecreasing process

$$G := \frac{1}{2}\int_0^{\cdot} (\rho_t, \mathrm{d}[S,S]_t \rho_t) \equiv \frac{1}{2}\int_0^{\cdot} \sum_{i=1}^d \sum_{j=1}^d \rho_t^i \rho_t^j \mathrm{d}[S^j, S^i]_t \tag{A.1}$$

is such that $\mathbb{P}[G_T < \infty] = 1$ holds for all $T \in \mathbb{R}_+$.

Under the validity of any of (A1), (B1), (C1), and with the above notation, it holds that

$$\log(\widehat{X}) = G + L, \quad \text{where } L := \int_0^{\cdot} \sum_{i=1}^{d} \rho_t^i \mathrm{d}M_t^i. \tag{A.2}$$

Furthermore, under the validity of any of the equivalent (A1), (B1), (C1), condition (A2) of Assumption 2.2 is equivalent to any of the following:

(B2) $\mathbb{P}[\lim_{t\to\infty} \widehat{X}_t = \infty] = 1$.

(C2) $\mathbb{P}[G_\infty = \infty] = 1$, *where* $G_\infty := \uparrow \lim_{t\to\infty} G_t$.

Proof. The fact that the three conditions (A1), (B1), and (C1) are equivalent, as well as the validity of (A.2), can be found in Kardaras (2010a, theorem 4). Now, assume any of the equivalent conditions (A1), (B1), or (C1). Clearly, (B2) implies (A2). On the other hand, suppose that there exists $X \in \mathcal{X}$ such that $\mathbb{P}[\lim_{t\to\infty} X_t = \infty] = 1$. The nonnegative supermartingale convergence theorem implies that $\lim_{t\to\infty}(X_t/\widehat{X}_t)$ $\mathbb{P}$-a.s. exists in $\mathbb{R}_+$, which implies that $\mathbb{P}[\lim_{t\to\infty} \widehat{X}_t = \infty] = 1$ holds as well. Therefore, (A2) implies (B2). Continuing, note that (A.2) implies that

$$[L, L] = \int_0^{\cdot} (\rho_t, \mathrm{d}[M, M]_t \rho_t) = \int_0^{\cdot} (\rho_t, \mathrm{d}[S, S]_t \rho_t) = 2G.$$

In view of the celebrated result of Dambis, Dubins, and Schwarz—see theorem 3.4.6 in Karatzas and Shreve (1991)—there exists a standard Brownian motion β (in a potentially enlarged probability space, and the Brownian motion property of β is with respect to its own natural filtration) such that $L_t = \beta_{2G_t}$ holds for $t \in \mathbb{R}_+$. It follows that $\log(\widehat{X}_t) = G_t + \beta_{2G_t}$ holds for $t \in \mathbb{R}_+$. Therefore, on $\{G_\infty < \infty\}$, $\lim_{t\to\infty} \widehat{X}_t$ a.s. exists and is $\mathbb{R}_+$-valued. On the other hand, the strong law of large numbers for Brownian motion implies that on $\{G_\infty = \infty\}$, $\lim_{t\to\infty}(\log(\widehat{X}_t)/G_t) = 1$ a.s. holds, which in turn implies that $\lim_{t\to\infty} \widehat{X}_t = \infty$ a.s. holds. The previous facts imply the a.s. set-equality $\{G_\infty = \infty\} = \{\lim_{t\to\infty} \widehat{X}_t = \infty\}$, which establishes the equivalence of conditions (B2) and (C2) and completes the proof. □

REMARK A.2. In Itô processes models, it holds that $B^i = \int_0^{\cdot} S_t^i b_t^i \mathrm{d}t$ and $M^i = \int_0^{\cdot} S_t^i \sum_{j=1}^{m} \sigma_t^{ij} \mathrm{d}W_t^j$ for $i \in \{1, \dots, d\}$, where $b = (b^1, \dots, b^d)$ is the predictable d-dimensional vector of excess rates of return, $(W^1, \dots, W^m)$ is an m-dimensional standard Brownian motion, and we write $c = \sigma\sigma^\top$ for the predictable $d \times d$ matrix-valued process of local covariances. According to Theorem A.1, condition (A1) of Assumption 2.2 is equivalent to the fact that there exists a d-dimensional process ρ such that $c\rho = b$, in which case we write $\rho = c^\dagger b$ where $c^\dagger$ is the Moore–Penrose pseudo-inverse of c, and that $G := (1/2)\int_0^{\cdot}(b_t, c_t^\dagger b_t)\mathrm{d}t = (1/2)\int_0^{\cdot}(\rho_t, c_t\rho_t)\mathrm{d}t$ is an a.s. finitely valued process. Observe that the process G is half of the integrated squared risk-premium in the market.

A.2. Proof of Proposition 2.6.

Since ${}^{\alpha}\widehat{X}/{}^{\alpha}\widehat{X}^* = \alpha + (1-\alpha)(\widehat{X}/\widehat{X}^*)$ holds in view of (2.4), we only need to establish that $0 = \liminf_{t\to\infty}(\widehat{X}_t/\widehat{X}_t^*) < \limsup_{t\to\infty}(\widehat{X}_t/\widehat{X}_t^*) = 1$. The fact that $\limsup_{t\to\infty}(\widehat{X}_t/\widehat{X}_t^*) = 1$ follows directly from $\lim_{t\to\infty} \widehat{X}_t = \infty$. On the other hand, the fact that $\liminf_{t\to\infty}(\widehat{X}_t/\widehat{X}_t^*) = 0$ follows immediately from the next result (which is stated separately as it is also used on another occasion) and the martingale version of the Borel–Cantelli lemma.

LEMMA A.3. *Let σ be a stopping time with $\mathbb{P}[\sigma < \infty] = 1$. For $\alpha \in (0, 1)$ define the stopping time $T := \inf\{t \in (\sigma, \infty) \mid \widehat{X}_t/\widehat{X}^*_t \leq \alpha\}$. Then $\mathbb{P}[T < \infty] = 1$.*

Proof. Recall that $\lim_{t\to\infty} \widehat{X}_t = \infty$ holds by Theorem A.1. Using the result of Dambis, Dubins and Schwarz—theorem 3.4.6 in Karatzas and Shreve (1991)—and a time-change argument, (A.2) implies that we can assume without loss of generality that $\widehat{X}$ satisfies $\widehat{X}_t = \exp(t/2 + \beta_t)$ for $t \in \mathbb{R}_+$, where β is a standard Brownian motion. Furthermore, using again the fact that $\lim_{t\to\infty} \widehat{X}_t = \infty$, we may assume without loss of generality that σ is a time of maximum of $\widehat{X}$. Then, the independent increments property of Brownian motion implies that we can additionally assume without loss of generality that $\sigma = 0$. Set $\sigma_0 = 0$ and, via induction, for each $n \in \mathbb{N}$ set

$$\sigma_n := \inf\left\{t \in (\sigma_{n-1}, \infty) \mid \widehat{X}_t = \mathrm{e}\widehat{X}_{\sigma_{n-1}}\right\}, \text{ and } T_n = \inf\left\{t \in (\sigma_{n-1}, \infty) \mid \widehat{X}_t/\widehat{X}^*_t = \alpha\right\}.$$

With $T = T_1$, we wish to show that $\mathbb{P}[T < \infty] = 1$. For each $n \in \mathbb{N}$, define the event $A_n := \{T_n < \sigma_n\}$. Note that $\mathbb{P}[A_n \mid \mathcal{F}_{\sigma_{n-1}}] = \mathbb{P}[A_1]$ holds for all $n \in \mathbb{N}$ in view of the regenerating property of Brownian motion and the fact that each $\sigma_{n-1}, n \in \mathbb{N}$, is a time of maximum of $\widehat{X}$. Since $\limsup_{n\to\infty} A_n \subseteq \{T < \infty\}$, the martingale version of the Borel–Cantelli lemma implies that $\mathbb{P}[T < \infty] = 1$ will be established as long as we can show that $\mathbb{P}[T_1 < \sigma_1] = \mathbb{P}[A_1] > 0$.

Since $\int_0^\infty \mathbb{I}_{\{\widehat{X}_t < \widehat{X}^*_t\}} \mathrm{d}\widehat{X}^*_t = 0$ a.s. holds, Itô's formula implies that

$$\frac{\widehat{X}^*}{\widehat{X}} = 1 + \int_0^\cdot \widehat{X}^*_t \mathrm{d}\left(\frac{1}{\widehat{X}_t}\right) + \log(\widehat{X}^*).$$

Both processes $\widehat{X}^*/\widehat{X}$ and $\log(\widehat{X}^*)$ are bounded on the stochastic interval $[0, \sigma_1 \wedge T_1]$—therefore, since $\mathbb{P}[\sigma_1 < \infty] = 1$ and $\int_0^\cdot \widehat{X}^*_t \mathrm{d}(1/\widehat{X}_t)$ is a local martingale (by Assumption 2.2 and the fact that $1 \in \mathcal{X}$), a localization argument gives

$$\mathbb{P}[\sigma_1 \leq T_1] + \frac{1}{\alpha}\mathbb{P}[T_1 < \sigma_1] = \mathbb{E}\left[\frac{\widehat{X}^*_{\sigma_1\wedge T_1}}{\widehat{X}_{\sigma_1\wedge T_1}}\right] = 1 + \mathbb{E}\left[\log\left(\widehat{X}^*_{\sigma_1\wedge T_1}\right)\right] \geq 1 + \mathbb{P}[\sigma_1 \leq T_1],$$

which gives $\mathbb{P}[T_1 < \sigma_1] \geq \alpha > 0$ and completes the proof of Lemma A.3. □

A.3. Proof of Theorem 3.5.

For the purposes of Subsection A.3, only condition (A1) of Assumption 2.2 is in force. Fix an a.s. finitely valued stopping time T throughout. As the result of Theorem 3.5 for the case $\alpha = 0$ is known, we tacitly assume that $\alpha \in (0, 1)$ throughout.

A.3.1. Existence. We shall first prove existence of a process with the numéraire property in ${}^\alpha\mathcal{X}$ for investment over the period $[0, T]$. As T is a.s. finitely valued, without loss of generality we shall assume that all processes that appear below are constant after time T, and their value after time T is equal to their value at time T. In particular, the limiting value of a process for time tending to infinity exists and is equal to its value at time T.

Define $\mathcal{X}^\circ$ as the class of all nonnegative càdlàg processes Y with $Y_0 \leq 1$ and with the property that YX is a supermartingale for all $X \in \mathcal{X}$. Note that $(1/\widehat{X}) \in \mathcal{X}^\circ$. In a similar way, define $\mathcal{X}^{\circ\circ}$ as the class of all nonnegative càdlàg processes χ with $\chi_0 \leq 1$ and with the property that $Y\chi$ is a supermartingale for all $Y \in \mathcal{X}^{\circ}$. It is clear that $\mathcal{X} \subseteq \mathcal{X}^{\circ\circ}$. The next result reveals the exact structure of $\mathcal{X}^{\circ\circ}$.

THEOREM A.4 (Optional Decomposition Theorem of Föllmer and Kramkov *1997*, Stricker and Yan *1998*). *The class $\mathcal{X}^{\circ\circ}$ consists exactly of all processes χ of the form $\chi = X(1 - A)$, where $X \in \mathcal{X}$ and A is an adapted, nonnegative and nondecreasing càdlàg process with* $0 \leq A \leq 1$.

The result that follows enables one to construct a process that will be a candidate to have the numéraire property in ${}^{\alpha}\mathcal{X}$ for investment over the interval $[0, T]$.

LEMMA A.5. *For any $\alpha \in [0, 1)$ and $t \in [0, \infty]$, the set $\{Z_t \mid Z \in {}^{\alpha}\mathcal{X}\}$ is convex and bounded in $\mathbb{P}$-measure, the latter meaning that* $\lim_{K\to\infty} \sup_{Z\in{}^{\alpha}\mathcal{X}} \mathbb{P}[Z_t > K] = 0$.

Proof. Fix $\alpha \in [0, 1)$. Let $\lambda \in [0, 1]$ and pick processes $X \in \mathcal{X}$ and $X' \in \mathcal{X}$. Since $\mathcal{X}$ is convex, $((1 - \lambda)^{\alpha}X + \lambda^{\alpha}X') \in \mathcal{X}$. Furthermore, since

$$\alpha((1 - \lambda)^{\alpha}X + \lambda^{\alpha}X')^* \leq (1 - \lambda)\alpha^{\alpha}X^* + \lambda\alpha({}^{\alpha}X')^* \leq \alpha((1 - \lambda)^{\alpha}X + \lambda^{\alpha}X'),$$

we obtain $((1 - \lambda)^{\alpha}X + \lambda^{\alpha}X') \in {}^{\alpha}\mathcal{X}$, which shows that ${}^{\alpha}\mathcal{X}$ is convex for all $\alpha \in [0, 1)$.

Furthermore, it holds that $\sup_{X\in\mathcal{X}} \mathbb{E}[X_\infty/\widehat{X}_\infty] \leq 1$ and, using Markov's inequality, we see that $\{X_\infty/\widehat{X}_\infty \mid X \in \mathcal{X}\}$ is bounded in $\mathbb{P}$-measure. Since $\mathbb{P}[\widehat{X}_\infty > 0] = 1$, the set $\{X_\infty \mid X \in \mathcal{X}\}$ is bounded in $\mathbb{P}$-measure; the same is then true for $\{{}^{\alpha}X_t \mid X \in \mathcal{X}\} \subseteq \{X_t \mid X \in \mathcal{X}\} \subseteq \{X_\infty \mid X \in \mathcal{X}\}$ for any value of $t \in [0, \infty]$. □

In the sequel, fix $\alpha \in (0, 1)$. In view of Lemma A.5 and theorem 1.1(4) in Kardaras (2010b), there exists a random variable $\check{\chi}_\infty$ in the closure in $\mathbb{P}$-measure of $\{X_\infty \mid X \in {}^{\alpha}\mathcal{X}\}$ such that $\mathbb{E}[X_\infty/\check{\chi}_\infty] \leq 1$ holds for all $X \in {}^{\alpha}\mathcal{X}$. Define the countable set $\mathbb{T} = \{k/2^m \mid k \in \mathbb{N},\ m \in \mathbb{N}\}$. A repeated application of lemma A1.1 in Delbaen and Schachermayer (1994) combined with Lemma A.5 and a diagonalization argument implies that one can find an ${}^{\alpha}\mathcal{X}$-valued sequence $(X^n)_{n\in\mathbb{N}}$ such that $\check{\chi}_\infty = \lim_{n\to\infty} X^n_\infty$ and $\lim_{n\to\infty} X^n_t$ a.s. exists simultaneously for all $t \in \mathbb{T}$. Define then $\check{\chi}_t = \lim_{n\to\infty} X^n_t$ for all $t \in \mathbb{T}$. Since T is a.s. finitely valued and all processes are constant after T, it is straightforward that $\check{\chi}_\infty = \lim_{t\to\infty} \check{\chi}_t$ a.s.

Since $\mathbb{E}[Y_t X^n_t \mid \mathcal{F}_s] \leq Y_s X^n_s$ holds for all $n \in \mathbb{N}$, $Y \in \mathcal{X}^{\circ}$, $t \in \mathbb{T}$ and $s \in \mathbb{T} \cap [0, t]$, the conditional version of Fatou's lemma gives that $\mathbb{E}[Y_t\check{\chi}_t \mid \mathcal{F}_s] \leq Y_s\check{\chi}_s$ holds for all $Y \in \mathcal{X}^{\circ}$, $t \in \mathbb{T}$ and $s \in \mathbb{T} \cap [0, t]$. In particular, with $\widehat{Y} := 1/\widehat{X} \in \mathcal{X}^{\circ}$, the process $(\widehat{Y}_t\check{\chi}_t)_{t\in\mathbb{T}}$ is a supermartingale in the corresponding stochastic basis with time-index $\mathbb{T}$. Since $\mathbb{P}[\inf_{s\in[0,t]} \widehat{Y}_s > 0] = 1$ holds for all $t \in \mathbb{R}_+$, the supermartingale convergence theorem implies that there exists a nonnegative càdlàg process χ such that[3] $\chi_s = \lim_{\mathbb{T}\ni t\downarrow\downarrow s} \check{\chi}_t$ holds for all $s \in \mathbb{R}_+$. (The notation "$\lim_{\mathbb{T}\ni t\downarrow\downarrow s}$" denotes limit along times $t \in \mathbb{T}$ that are *strictly* greater than $s \in \mathbb{R}_+$ and converge to s.) The fact that $\mathbb{E}[Y_t\check{\chi}_t \mid \mathcal{F}_s] \leq Y_s\check{\chi}_s$ holds for all $Y \in \mathcal{X}^{\circ}$, $t \in \mathbb{T}$ and $s \in \mathbb{T} \cap [0, t]$, right-continuity of the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and the conditional version of Fatou's lemma give that $\mathbb{E}[Y_t\chi_t \mid \mathcal{F}_s] \leq Y_s\chi_s$ holds for all $Y \in \mathcal{X}^{\circ}$, $t \in \mathbb{R}_+$ and $s \in [0, t]$. Therefore, $\chi \in \mathcal{X}^{\circ\circ}$. Of course, $\chi_\infty = \check{\chi}_\infty = \lim_{t\to\infty} \chi_t$ a.s. holds. In view of Theorem A.4, it holds that $\chi = \widetilde{X}(1 - A)$, where $\widetilde{X} \in \mathcal{X}$ and A is an adapted, nonnegative and nondecreasing càdlàg process with $0 \leq A \leq 1$. Furthermore, note that $\mathbb{E}[X_\infty/\chi_\infty] \leq 1$ holds for all $X \in {}^{\alpha}X$.

[3]Note that χ is indeed the limit of $(X^n)_{n\in\mathbb{N}}$ in the "Fatou" sense. Fatou-convergence has proved to be extremely useful in the theory of Mathematical Finance; for example, see Föllmer and Kramkov (1997), Kramkov and Schachermayer (1999), and Žitković (2002).

Continuing, we shall show that $A \equiv 0$ and $\chi(= \widetilde{X}) \in {}^{\alpha}\mathcal{X}$. If $(X^n)_{n \in \mathbb{N}}$ is the ${}^{\alpha}\mathcal{X}$-valued sequence such that $\check{\chi}_t = \lim_{n\to\infty} X^n_t$ holds a.s. simultaneously for all $t \in \mathbb{T}$, we have that $X^n_t \geq \alpha X^n_s$ a.s. holds for all $t \in \mathbb{T}$ and $s \in \mathbb{T} \cap [0, t]$. By passing to the limit, and using the fact that $\mathbb{T}$ is countable, we obtain that $\check{\chi}_t \geq \alpha \check{\chi}_s$ holds a.s. simultaneously for all $t \in \mathbb{T}$ and $s \in \mathbb{T} \cap [0, t]$. Therefore, $\chi_t \geq \alpha \chi_s$ holds a.s. simultaneously for all $t \in \mathbb{R}_+$ and $s \in [0, t]$. Then,

$$\widetilde{X}_t = \frac{\chi_t}{1 - A_t} \geq \frac{\chi_t}{1 - A_s} \geq \alpha \frac{\chi_s}{1 - A_s} = \alpha \widetilde{X}_s$$

holds a.s. simultaneously for all $t \in \mathbb{R}_+$ and $s \in [0, t]$. It follows that $\widetilde{X} \in {}^{\alpha}\mathcal{X}$. This implies, in particular, that $\mathbb{E}[\widetilde{X}_\infty / \chi_\infty] \leq 1$ has to hold. Since $\widetilde{X}_\infty / \chi_\infty = 1/(1 - A_\infty) \geq 1$, we obtain $\mathbb{P}[A_\infty = 0] = 1$, i.e., $A \equiv 0$. Therefore, $\chi = \widetilde{X}$ and $\mathbb{E}[X_\infty / \widetilde{X}_\infty] \leq 1$ holds for all $X \in {}^{\alpha}X$, which concludes the proof of existence of a wealth process that possesses the numéraire property in ${}^{\alpha}\mathcal{X}$ for investment over $[0, T]$.

A.3.2. Uniqueness. We proceed in establishing uniqueness of a process with the numéraire property in ${}^{\alpha}\mathcal{X}$ for investment over the period $[0, T]$. We start by stating and proving a result that will be used again later.

LEMMA A.6. *Let $Z \in {}^{\alpha}X$, and let σ be a stopping time such that $Z_\sigma = Z^*_\sigma$ a.s. holds on $\{\sigma < \infty\}$. Fix $X \in {}^{\alpha}\mathcal{X}$ and $A \in \mathcal{F}_\sigma$ and define a new process*[4] $\xi := Z\mathbb{I}_{[0,\sigma[} + (Z\mathbb{I}_{\Omega \setminus A} + (Z_\sigma / X_\sigma) X \mathbb{I}_A)\mathbb{I}_{[\sigma,\infty[}$. *Then,* $\xi \in {}^{\alpha}\mathcal{X}$.

Proof. It is straightforward to check that $\xi \in \mathcal{X}$. To see that $\xi \in {}^{\alpha}\mathcal{X}$, note that $\xi / \xi^* = Z/Z^* \geq \alpha$ holds on $[\![0, \sigma[\![\cup ([\![\sigma, \infty[\![\cap (\Omega \setminus A))$, while, using the fact that $\xi^*_\sigma = Z^*_\sigma = Z_\sigma$ holds a.s.on $\{\sigma < \infty\}$,

$$\frac{\xi}{\xi^*} = \frac{X}{\sup_{t \in [\sigma, \cdot]} X_t} \geq \frac{X}{X^*} \geq \alpha, \quad \text{holds on } [\![\sigma, \infty[\![\cap A.$$

The result immediately follows. □

REMARK A.7. As can be seen via the use of simple counterexamples, if one drops the assumption that σ is a time of maximum of Z in the statement of Lemma A.6, the resulting process ξ may fail to satisfy the drawdown constraints. This is in direct contrast with the nonconstrained case $\alpha = 0$, where any stopping time σ will result in ξ being an element of $\mathcal{X}$. It is exactly this fact, a consequence of the path-dependent structure of the drawdown constraints, which results in portfolios with the numéraire property that depend on the investment horizon.

LEMMA A.8. *Let $Z \in {}^{\alpha}\mathcal{X}$ be such that $\mathrm{Err}_T(X|Z) \leq 0$ holds for all $X \in {}^{\alpha}X$, and suppose that σ is a stopping time such that $Z_\sigma = Z^*_\sigma$ a.s. holds on $\{\sigma < \infty\}$. Then,*

$$\mathbb{E}\left[\frac{X_T}{Z_T} \;\middle|\; \mathcal{F}_{T \wedge \sigma} \right] \leq \frac{X_{T\wedge\sigma}}{Z_{T\wedge\sigma}} \text{ holds a.s. for all } X \in {}^{\alpha}\mathcal{X}.$$

[4]Note that, since we tacitly assume that $\alpha \in (0, 1)$, $X_\sigma > 0$ a.s. holds on $\{\sigma < \infty\}$. Therefore, the process ξ is well defined.

Proof. Fix $X \in {}^{\alpha}\mathcal{X}$ and $A \in \mathcal{F}_\sigma$. Define the process $\xi := Z\mathbb{I}_{[0,\sigma[} + (Z I_{\Omega \setminus A} + (Z_\sigma / X_\sigma) X \mathbb{I}_A)\mathbb{I}_{[\sigma,\infty[}$; by Lemma A.6, $\xi \in {}^{\alpha}\mathcal{X}$. Furthermore, it is straightforward to check that

$$\frac{\xi_T}{Z_T} = \mathbb{I}_{\Omega \setminus (A \cap \{\sigma \le T\})} + \left(\frac{X_T}{Z_T} \frac{Z_\sigma}{X_\sigma} \right) \mathbb{I}_{A \cap \{\sigma \le T\}}.$$

Therefore, the fact that $\mathbb{E}\mathrm{rr}_T(\xi \,|\, Z) \le 0$ holds implies

$$\mathbb{E}\left[\frac{X_T}{Z_T} \frac{Z_\sigma}{X_\sigma} \mathbb{I}_{A \cap \{\sigma \le T\}} \right] \le \mathbb{P}[A \cap \{\sigma \le T\}].$$

As the previous is true for all $A \in \mathcal{F}_\sigma$, we obtain that $\mathbb{E}[X_T / Z_T \mid \mathcal{F}_\sigma] \le X_\sigma / Z_\sigma$ holds a.s. on $\{\sigma \le T\}$ for all $X \in {}^{\alpha}\mathcal{X}$. Combined with the fact that $\mathbb{E}[X_T/Z_T \mid \mathcal{F}_T] = X_T/Z_T$ trivially holds a.s. on $\{\sigma > T\}$ for all $X \in {}^{\alpha}\mathcal{X}$, we obtain the result. □

We now proceed to the actual proof of uniqueness. Assume that both $\widetilde{Z} \in {}^{\alpha}\mathcal{X}$ and $\widetilde{X} \in {}^{\alpha}\mathcal{X}$ have the numéraire property in ${}^{\alpha}\mathcal{X}$ for investment over $[0, T]$. Since $\mathbb{P}[T < \infty] = 1$, Proposition 3.3 implies that $\mathbb{P}[\widetilde{X}_T = \widetilde{Z}_T] = 1$. We shall show below that $\widetilde{Z} \le \widetilde{X}$ holds on $[0, T]$. Interchanging the roles of $\widetilde{X}$ and $\widetilde{Z}$, it will also follow that $\widetilde{X} \le \widetilde{Z}$ holds on $[0, T]$, which will establish that $\widetilde{X} = \widetilde{Z}$ holds on $[0, T]$ and will complete the proof of Theorem 3.5.

Since $\mathbb{P}[\widetilde{X}_T = \widetilde{Z}_T] = 1$ and $\mathbb{E}\mathrm{rr}_T(X | \widetilde{Z}) \le 0$ holds for all $X \in {}^{\alpha}\mathcal{X}$, Lemma A.8 implies that $1 = \mathbb{E}[\widetilde{X}_T / \widetilde{Z}_T \mid \mathcal{F}_{T \wedge \sigma}] \le \widetilde{X}_{T \wedge \sigma} / \widetilde{Z}_{T \wedge \sigma}$ a.s. holds whenever σ is a stopping time such that $\widetilde{Z}_\sigma = \widetilde{Z}^*_\sigma$ a.s. holds on $\{\sigma < \infty\}$. The fact that $\widetilde{Z}_{T \wedge \sigma} \le \widetilde{X}_{T \wedge \sigma}$ a.s. holds whenever σ is a stopping time such that $\widetilde{Z}_\sigma = \widetilde{Z}^*_\sigma$ a.s. holds on $\{\sigma < \infty\}$ implies in a straightforward way that $\widetilde{Z}^* \le \widetilde{X}^*$ holds on $[0, T]$.

We now claim that $\mathbb{P}[\widetilde{Z}_T = \widetilde{X}_T] = 1$ combined with $\widetilde{Z}^* \le \widetilde{X}^*$ holding on $[0, T]$ imply that $\widetilde{Z} \le \widetilde{X}$ on $[0, T]$, which will complete the proof. To see the last claim, for $\epsilon > 0$ define the stopping time

$$T_\epsilon := \inf \left\{ t \in \mathbb{R}_+ \mid \widetilde{Z}_t > (1 + \epsilon)\widetilde{X}_t \right\}.$$

We shall show that $\mathbb{P}[T_\epsilon < T] = 0$; as this will hold for all $\epsilon > 0$, it will follow that $\widetilde{Z} \le \widetilde{X}$ holds on $[0, T]$. Define a new process $\widetilde{X}^\epsilon$ via

$$\widetilde{X}^\epsilon = \widetilde{Z}\mathbb{I}_{[0,T_\epsilon[} + \left(\frac{\widetilde{Z}_{T^\epsilon}}{\widetilde{X}_{T^\epsilon}} \right) \widetilde{X}\mathbb{I}_{[T^\epsilon,\infty[} = \widetilde{Z}\mathbb{I}_{[0,T_\epsilon[} + (1 + \epsilon)\,\widetilde{X}\mathbb{I}_{[T^\epsilon,\infty[}.$$

We first show that $\widetilde{X}^\epsilon \in {}^{\alpha}\mathcal{X}$. The fact that $\widetilde{X} \in \mathcal{X}$ is obvious. Note also that $\widetilde{X}^\epsilon \ge \alpha(\widetilde{X}^\epsilon)^*$ clearly holds on $[\![0, T^\epsilon[\![$, since $\widetilde{Z} \in {}^{\alpha}X$. On the other hand,

$$(\widetilde{X}^\epsilon)^*_t = (\widetilde{Z}_{T^\epsilon})^* \vee \sup_{s \in [T^\epsilon, t]} \left((1 + \epsilon)\widetilde{X}_t \right) \le (1 + \epsilon)\widetilde{X}^*_t \text{ holds for } t \ge T^\epsilon,$$

the latter inequality holding in view of the fact that $\widetilde{Z}^* \le \widetilde{X}^*$. Therefore, for $t \ge T^\epsilon$ it holds that $\widetilde{X}^\epsilon_t = (1 + \epsilon)\widetilde{X}_t \ge (1 + \epsilon)\alpha\widetilde{X}^*_t \ge \alpha(\widetilde{X}^\epsilon)^*_t$. It follows that $\widetilde{X}^\epsilon \ge \alpha(\widetilde{X}^\epsilon)^*$ also holds on $[\![T^\epsilon, \infty[\![$, which shows that $\widetilde{X}^\epsilon \in {}^{\alpha}\mathcal{X}$. Note that

$$\widetilde{X}^\epsilon_T = \widetilde{Z}_T\mathbb{I}_{\{T < T^\epsilon\}} + (1 + \epsilon)\widetilde{X}_T\mathbb{I}_{\{T^\epsilon \le T\}} = \widetilde{X}_T\mathbb{I}_{\{T < T^\epsilon\}} + (1 + \epsilon)\widetilde{X}_T\mathbb{I}_{\{T^\epsilon \le T\}},$$

which implies that $\widetilde{X}^\epsilon_T / \widetilde{X}_T = 1 + \epsilon\mathbb{I}_{\{T^\epsilon \le T\}}$ and, as a consequence, $\mathbb{E}\mathrm{rr}_T(\widetilde{X}^\epsilon | \widetilde{X}) = \epsilon\mathbb{P}[T^\epsilon \le T]$. In case $\mathbb{P}\,[T^\epsilon \le T] > 0$, it would follow that $\widetilde{X}$ fails to have the numéraire property in

${}^{\alpha}\mathcal{X}$ for investment in $[0, T]$. Therefore, $\mathbb{P}[T^{\epsilon} \leq T] = 0$, which implies that $\widetilde{Z} \leq \widetilde{X}$ holds on $[0, T]$, as already mentioned. The proof of Theorem 3.5 is complete.

A.4. *Proof of Theorem 3.8.*

The main tool toward proving assertion (1) of Theorem 3.8 is the following auxiliary result.

LEMMA A.9. *For any $X \in \mathcal{X}$, $\lim_{t\to\infty}(X^*_t/\widehat{X}^*_t)$ a.s. exists. Moreover, it a.s. holds that*

$$\lim_{t\to\infty}\left(\frac{X^*_t}{\widehat{X}^*_t}\right) = \lim_{t\to\infty}\left(\frac{X_t}{\widehat{X}_t}\right).$$

Proof. For $t \in \mathbb{R}_+$, define the $[0, t]$-valued random time $\widehat{\rho}_t := \sup\{s \in [0, t] \mid \widehat{X}_s = \widehat{X}^*_s\}$; then, $\widehat{X}^*_t = \widehat{X}_{\widehat{\rho}_t}$. Note that $\mathbb{P}[\uparrow \lim_{t\to\infty} \widehat{\rho}_t = \infty] = 1$ holds in view of Assumption 2.2. It follows that, for any $X \in \mathcal{X}$, it a.s. holds that

$$\liminf_{t\to\infty}\left(\frac{X^*_t}{\widehat{X}^*_t}\right) = \liminf_{t\to\infty}\left(\frac{X^*_t}{\widehat{X}^*_{\widehat{\rho}_t}}\right) \geq \liminf_{t\to\infty}\left(\frac{X_{\widehat{\rho}_t}}{\widehat{X}_{\widehat{\rho}_t}}\right) = \lim_{t\to\infty}\left(\frac{X_t}{\widehat{X}_t}\right). \tag{A.3}$$

In what follows, fix $X \in \mathcal{X}$. For $t \in \mathbb{R}_+$ define $\rho_t := \sup\{s \in [0, t] \mid X_s = X^*_s\}$, which is a $[0, t]$-valued random time. For each $t \in \mathbb{R}_+$, $X^*_t = X_{\rho_t}$. Note that the set-inclusions $\{\uparrow \lim_{t\to\infty} \rho_t < \infty\} \subseteq \{\sup_{t\in\mathbb{R}_+} X_t < \infty\} \subseteq \{\mathrm{rr}_\infty(X|\widehat{X}) = -1\}$ are valid a.s., the last in view of Assumption 2.2. Therefore,

$$\lim_{t\to\infty}\left(\frac{X_t}{\widehat{X}_t}\right) = \lim_{t\to\infty}\left(\frac{X^*_t}{\widehat{X}^*_t}\right) = 0 \text{ holds on } \left\{\lim_{t\to\infty} \rho_t < \infty\right\}. \tag{A.4}$$

Furthermore,

$$\limsup_{t\to\infty}\left(\frac{X^*_t}{\widehat{X}^*_t}\right) = \limsup_{t\to\infty}\left(\frac{X^*_{\rho_t}}{\widehat{X}^*_t}\right) \leq \limsup_{t\to\infty}\left(\frac{X_{\rho_t}}{\widehat{X}_{\rho_t}}\right) = \lim_{t\to\infty}\left(\frac{X_t}{\widehat{X}_t}\right) \text{ holds on } \left\{\lim_{t\to\infty} \rho_t = \infty\right\}. \tag{A.5}$$

The claim now readily follows from (A.3), (A.4), and (A.5). □

Proof of Theorem 3.8, statement (1) In the sequel, fix $X \in \mathcal{X}$ and assume that $\alpha \in (0, 1)$. Results for the case $\alpha = 0$ are well understood and not discussed.

To ease notation, let $D := X/X^*$ and $\widehat{D} := \widehat{X}/\widehat{X}^*$. The process D is $[0, 1]$-valued and $\widehat{D}$ is $(0, 1]$-valued. Observe that

$$\frac{{}^{\alpha}X}{{}^{\alpha}\widehat{X}} = \frac{\alpha(X^*)^{1-\alpha} + \alpha(X^*)^{-\alpha}X}{\alpha(\widehat{X}^*)^{1-\alpha} + \alpha(\widehat{X}^*)^{-\alpha}\widehat{X}} = \left(\frac{X^*}{\widehat{X}^*}\right)^{1-\alpha}\left(\frac{\alpha + (1-\alpha)D}{\alpha + (1-\alpha)\widehat{D}}\right).$$

In view of Lemma A.9, $\lim_{t\to\infty}(X^*_t/\widehat{X}^*_t)^{1-\alpha} = (1 + \mathrm{rr}_\infty(X|\widehat{X}))^{1-\alpha}$ holds. First, the fact that

$$\frac{\alpha + (1-\alpha)D}{\alpha + (1-\alpha)\widehat{D}} \leq \frac{1}{\alpha}$$

implies that ${}^{\alpha}X/{}^{\alpha}\widehat{X} \le (1/\alpha)(X^*/\widehat{X}^*)^{1-\alpha}$, which readily gives (3.3) on $\{\mathrm{rr}_\infty(X|\widehat{X}) = -1\}$. Furthermore, the facts that $0 \le D \le 1$, $0 < \widehat{D} \le 1$ and $\lim_{t\to\infty}(D_t/\widehat{D}_t) = 1$, the latter holding a.s. on $\{\mathrm{rr}_\infty(X|\widehat{X}) > -1\}$ in view of Lemma A.9, imply that

$$\limsup_{t\to\infty}\left|\frac{\alpha + (1-\alpha)D_t}{\alpha + (1-\alpha)\widehat{D}_t} - 1\right| \le \frac{1-\alpha}{\alpha}\limsup_{t\to\infty}\left|D_t - \widehat{D}_t\right| = 0 \text{ holds on } \left\{\mathrm{rr}_\infty(X|\widehat{X}) > -1\right\}.$$

Therefore, $\lim_{t\to\infty}({}^{\alpha}X_t/{}^{\alpha}\widehat{X}_t) = (1 + \mathrm{rr}_\infty(X|\widehat{X}))^{1-\alpha}$ also holds on the event $\{\mathrm{rr}_\infty(X|\widehat{X}) > -1\}$, which completes the proof of statement (1) of Theorem 3.8. □

Proof of Theorem 3.8, statement (2). Let τ be a time of maximum of $\widehat{X}$. Recall the definition of the stopping times $(\tau_\ell)_{\ell\in\mathbb{R}_+}$ from (3.5). In view of statement (1) of Theorem 3.8,

$$\mathrm{rr}_\tau({}^{\alpha}X|{}^{\alpha}\widehat{X}) = \lim_{\ell\to\infty}\left(\frac{{}^{\alpha}X_{\tau\wedge\tau_\ell}}{{}^{\alpha}\widehat{X}_{\tau\wedge\tau_\ell}}\right) - 1 \tag{A.6}$$

a.s. holds. Now, observe that $\tau\wedge\tau_\ell$ is a time of maximum of $\widehat{X}$ for each $\ell\in\mathbb{R}_+$; therefore, ${}^{\alpha}\widehat{X}_{\tau\wedge\tau_\ell} = (\widehat{X}_{\tau\wedge\tau_\ell})^{1-\alpha} = (\widehat{X}^*_{\tau\wedge\tau_\ell})^{1-\alpha}$. It then follows that

$$\frac{{}^{\alpha}X_{\tau\wedge\tau_\ell}}{{}^{\alpha}\widehat{X}_{\tau\wedge\tau_\ell}} = \alpha\left(\frac{X^*_{\tau\wedge\tau_\ell}}{\widehat{X}^*_{\tau\wedge\tau_\ell}}\right)^{1-\alpha} + (1-\alpha)\left(\frac{X_{\tau\wedge\tau_\ell}}{\widehat{X}_{\tau\wedge\tau_\ell}}\right)\left(\frac{X^*_{\tau\wedge\tau_\ell}}{\widehat{X}^*_{\tau\wedge\tau_\ell}}\right)^{-\alpha}. \tag{A.7}$$

Define $\chi := X/\widehat{X}$ and, in the obvious way, $\chi^* := \sup_{t\in[0,\cdot]}(X_t/\widehat{X}_t)$. For $y\in\mathbb{R}_+$, the function $[y,\infty)\ni z\mapsto \alpha z^{1-\alpha} + (1-\alpha)yz^{-\alpha}$ is nondecreasing, which can be shown upon simple differentiation. With $y = \chi_{\tau\wedge\tau_\ell}$, $z_1 = X^*_{\tau\wedge\tau_\ell}/\widehat{X}^*_{\tau\wedge\tau_\ell} = X^*_{\tau\wedge\tau_\ell}/\widehat{X}_{\tau\wedge\tau_\ell} \ge y$ and $z_2 = \chi^*_{\tau\wedge\tau_\ell} \ge X^*_{\tau\wedge\tau_\ell}/\widehat{X}^*_{\tau\wedge\tau_\ell} = z_1$, (A.7) then implies that

$$\frac{{}^{\alpha}X_{\tau\wedge\tau_\ell}}{{}^{\alpha}\widehat{X}_{\tau\wedge\tau_\ell}} \le \alpha\left(\chi^*_{\tau\wedge\tau_\ell}\right)^{1-\alpha} + (1-\alpha)\chi_{\tau\wedge\tau_\ell}\left(\chi^*_{\tau\wedge\tau_\ell}\right)^{-\alpha}.$$

Define the process $\phi := \alpha(\chi^*)^{1-\alpha} + (1-\alpha)\chi(\chi^*)^{-\alpha}$; then, by the last estimate and (A.6),

$$\mathrm{rr}_\tau({}^{\alpha}X|{}^{\alpha}\widehat{X}) \le \liminf_{\ell\to\infty}(\phi_{\tau\wedge\tau_\ell}) - 1.$$

Since $\int_0^\infty \mathbb{I}_{\{\chi_t<\chi_t^*\}}\mathrm{d}\chi_t^* = 0$ a.s. holds, a straightforward use of Itô's formula gives

$$\phi = 1 + \int_0^\cdot (1-\alpha)\left(\chi_t^*\right)^{-\alpha}\mathrm{d}\chi_t;$$

since χ is a local martingale, ϕ is a local martingale as well. Since ϕ is nonnegative, it is a supermartingale with $\phi_0 = 1$, which implies that $\mathbb{E}\left[\phi_{\tau\wedge\tau_\ell}\right] \le 1$ holds for all $\ell\in\mathbb{R}_+$. It follows that

$$\mathbb{E}\mathrm{rr}_\tau({}^{\alpha}X|{}^{\alpha}\widehat{X}) = \mathbb{E}\left[\mathrm{rr}_\tau({}^{\alpha}X|{}^{\alpha}\widehat{X})\right] \le \mathbb{E}\left[\liminf_{\ell\to\infty}(\phi_{\tau\wedge\tau_\ell})\right] - 1 \le \liminf_{\ell\to\infty}\left(\mathbb{E}\left[\phi_{\tau\wedge\tau_\ell}\right]\right) - 1 \le 0.$$

Now, let σ be a time of maximum of $\widehat{X}$ with $\sigma \le \tau$. Fix $X\in\mathcal{X}$ and $A\in\mathcal{F}_\sigma$; by Lemma A.6, the process $\xi := {}^{\alpha}\widehat{X}\mathbb{I}_{[0,\sigma[} + ({}^{\alpha}\widehat{X}\mathbb{I}_{\Omega\setminus A} + {}^{\alpha}\widehat{X}_\sigma({}^{\alpha}X/{}^{\alpha}X_\sigma)\mathbb{I}_A)\mathbb{I}_{[\sigma,\infty[}$ is an element of ${}^{\alpha}\mathcal{X}$. Furthermore, it is straightforward to check that

$$\mathrm{rr}_\tau(\xi|{}^{\alpha}\widehat{X}) = \left(\frac{1 + \mathrm{rr}_\tau({}^{\alpha}X|{}^{\alpha}\widehat{X})}{1 + \mathrm{rr}_\sigma({}^{\alpha}X|{}^{\alpha}\widehat{X})} - 1\right)\mathbb{I}_{A\cap\{\sigma<\infty\}}.$$

Since $\mathbb{E}\mathrm{rr}_\tau(\xi\,|{}^\alpha\widehat{X}) \le 0$ has to hold by the result previously established, we obtain

$$\mathbb{E}\left[\frac{1+\mathrm{rr}_\tau\left({}^\alpha X|{}^\alpha\widehat{X}\right)}{1+\mathrm{rr}_\sigma\left({}^\alpha X|{}^\alpha\widehat{X}\right)}\mathbb{I}_{A\cap\{\sigma<\infty\}}\right] \le \mathbb{P}[A\cap\{\sigma<\infty\}].$$

Since the previous holds for all $A \in \mathcal{F}_\sigma$, we obtain that $\mathbb{E}[\mathrm{rr}_\tau({}^\alpha X|{}^\alpha\widehat{X}) \mid \mathcal{F}_\sigma] \le \mathrm{rr}_\sigma({}^\alpha X|{}^\alpha\widehat{X})$ holds on $\{\sigma<\infty\}$. On $\{\sigma=\infty\}$, we have $\sigma=\tau$ and $\mathbb{E}[\mathrm{rr}_\tau({}^\alpha X|{}^\alpha\widehat{X}) \mid \mathcal{F}_\sigma] = \mathrm{rr}_\infty({}^\alpha X|{}^\alpha\widehat{X}) = \mathrm{rr}_\sigma({}^\alpha X|{}^\alpha\widehat{X})$. Therefore, $\mathbb{E}[\mathrm{rr}_\tau({}^\alpha X|{}^\alpha\widehat{X}) \mid \mathcal{F}_\sigma] \le \mathrm{rr}_\sigma({}^\alpha X|{}^\alpha\widehat{X})$ holds. □

A.5. *Proof of Theorem 4.7.*

In the setting of Definition 4.5, consider a sequence $(\xi^n)_{n\in\mathbb{N}}$ of semimartingales and another semimartingale ξ. It is straightforward to check that $\mathcal{S}_{\mathrm{loc}}\text{-}\lim_{n\to\infty}\xi^n = \xi$ holds if and only if there exists a nondecreasing sequence $(\tau_k)_{k\in\mathbb{N}}$ of finitely valued stopping times with $\mathbb{P}[\lim_{k\to\infty}\tau_k=\infty]=1$ such that $\mathcal{S}_{\tau_k}\text{-}\lim_{n\to\infty}\xi^n=\xi$ holds for all $k\in\mathbb{N}$. For the proof of Theorem 4.7, we shall use the previous observation along the sequence $(\tau_\ell)_{\ell\in\mathbb{N}}$ of finitely valued stopping times defined in (3.5). Therefore, in the course of the proof, we keep $\ell\in\mathbb{R}_+$ fixed and will show that $\mathcal{S}_{\tau_\ell}\text{-}\lim_{n\to\infty}{}^\alpha\widetilde{X}^n = {}^\alpha\widehat{X}$.

As a first step, we shall show that $\mathbb{P}\text{-}\lim_{n\to\infty}{}^\alpha\widetilde{X}^n_{\tau_\ell} = {}^\alpha\widehat{X}_{\tau_\ell}$, where "$\mathbb{P}$-lim" denotes limit in probability. For each $n\in\mathbb{N}$, consider the process $\xi^n := {}^\alpha\widehat{X}\,\mathbb{I}_{[0,\tau_\ell[} + {}^\alpha\widehat{X}_{\tau_\ell}\left({}^\alpha\widetilde{X}^n/{}^\alpha\widetilde{X}^n_{\tau_\ell}\right)\mathbb{I}_{[\tau_\ell,\infty[}$. By Lemma A.6, $\xi^n\in{}^\alpha\mathcal{X}$ for all $n\in\mathbb{N}$. Furthermore, note that

$$\mathrm{rr}_{T_n}(\xi^n|{}^\alpha\widetilde{X}) = \mathrm{rr}_{\tau_\ell}\left({}^\alpha\widehat{X}|{}^\alpha\widetilde{X}^n\right)\mathbb{I}_{\{\tau_\ell<T_n\}} + \mathrm{rr}_{T_n}\left({}^\alpha\widehat{X}|{}^\alpha\widetilde{X}^n\right)\mathbb{I}_{\{T_n\le\tau_\ell\}} = \mathrm{rr}_{T_n\wedge\tau_\ell}\left({}^\alpha\widehat{X}|{}^\alpha\widetilde{X}^n\right).$$

Using the previous relationship, the assumptions of Theorem 4.7 give $\mathbb{E}\mathrm{rr}_{T_n\wedge\tau_\ell}({}^\alpha\widehat{X}|{}^\alpha\widetilde{X}^n)\le 0$ for all $n\in\mathbb{N}$. Furthermore, by Theorem 3.8, $\mathbb{E}\mathrm{rr}_{\tau_\ell}({}^\alpha\widetilde{X}^n|{}^\alpha\widehat{X})\le 0$ holds for all $n\in\mathbb{N}$. Therefore, $\mathbb{E}[\mathrm{rr}_{T_n\wedge\tau_\ell}({}^\alpha\widehat{X}|{}^\alpha\widetilde{X}^n)+\mathrm{rr}_{\tau_\ell}({}^\alpha\widetilde{X}^n|{}^\alpha\widehat{X})]\le 0$ holds for all $n\in\mathbb{N}$. Observe that the equality $\mathrm{rr}_{T_n\wedge\tau_\ell}({}^\alpha\widehat{X}|{}^\alpha\widetilde{X}^n)+\mathrm{rr}_{\tau_\ell}({}^\alpha\widetilde{X}^n|{}^\alpha\widehat{X}) = ({}^\alpha\widetilde{X}^n_{\tau_\ell}-{}^\alpha\widehat{X}_{\tau_\ell})^2/({}^\alpha\widehat{X}_{\tau_\ell}{}^\alpha\widetilde{X}^n_{\tau_\ell})$ holds on $\{\tau_\ell<T_n\}$, and that the inequality $\mathrm{rr}_{T_n\wedge\tau_\ell}({}^\alpha\widehat{X}|{}^\alpha\widetilde{X}^n)+\mathrm{rr}_{\tau_\ell}({}^\alpha\widetilde{X}^n|{}^\alpha\widehat{X})\ge -2$ is always true; therefore,

$$\mathrm{rr}_{T_n\wedge\tau_\ell}\left({}^\alpha\widehat{X}|{}^\alpha\widetilde{X}^n\right)+\mathrm{rr}_{\tau_\ell}\left({}^\alpha\widetilde{X}^n|{}^\alpha\widehat{X}\right) \ge \frac{\left({}^\alpha\widetilde{X}^n_{\tau_\ell}-{}^\alpha\widehat{X}_{\tau_\ell}\right)^2}{{}^\alpha\widehat{X}_{\tau_\ell}{}^\alpha\widetilde{X}^n_{\tau_\ell}}\mathbb{I}_{\{\tau_\ell<T_n\}} - 2\mathbb{I}_{\{T_n\le\tau_\ell\}}.$$

Since $\mathbb{E}[\mathrm{rr}_{T_n\wedge\tau_\ell}({}^\alpha\widehat{X}|{}^\alpha\widetilde{X}^n)+\mathrm{rr}_{\tau_\ell}({}^\alpha\widetilde{X}^n|{}^\alpha\widehat{X})]\le 0$ holds for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}\mathbb{P}[T_n\le\tau_\ell]=0$ holds in view of Theorem A.1, we obtain that

$$\lim_{n\to\infty}\mathbb{E}\left[\frac{\left({}^\alpha\widetilde{X}^n_{\tau_\ell}-{}^\alpha\widehat{X}_{\tau_\ell}\right)^2}{{}^\alpha\widehat{X}_{\tau_\ell}{}^\alpha\widetilde{X}^n_{\tau_\ell}}\mathbb{I}_{\{\tau_\ell<T_n\}}\right]=0.$$

Using again the fact that $\lim_{n\to\infty}\mathbb{P}[\tau_\ell<T_n]=1$, we obtain that $\mathbb{P}\text{-}\lim_{n\to\infty}{}^\alpha\widetilde{X}^n_{\tau_\ell}={}^\alpha\widehat{X}_{\tau_\ell}$.

Given $\mathbb{P}\text{-}\lim_{n\to\infty}{}^\alpha\widetilde{X}^n_{\tau_\ell}={}^\alpha\widehat{X}_{\tau_\ell}$, we now proceed in showing that $\mathbb{P}\text{-}\lim_{n\to\infty}(\widetilde{X}^n_{\tau_\ell}/\widehat{X}_{\tau_\ell})=1$. We use some arguments similar to the first part of the proof of statement (2) of Theorem 3.8, where the reader is referred to for certain details that are omitted here. Define $\chi^n := \widetilde{X}^n/\widehat{X}$ and $(\chi^n)^* := \sup_{t\in[0,\cdot]}(\widetilde{X}^n_t/\widehat{X}_t)$. It then follows that

$$\frac{{}^\alpha\widetilde{X}^n_{\tau_\ell}}{{}^\alpha\widehat{X}_{\tau_\ell}} \le \alpha\left((\chi^n)^*_{\tau_\ell}\right)^{1-\alpha}+(1-\alpha)\chi^n_{\tau_\ell}\left((\chi^n)^*_{\tau_\ell}\right)^{-\alpha} =: \phi^n_{\tau_\ell}, \tag{A.8}$$

where the process $\phi^n := \alpha\left((\chi^n)^*\right)^{1-\alpha} + (1-\alpha)\chi^n\left((\chi^n)^*\right)^{-\alpha}$ is a nonnegative local martingale for each $n \in \mathbb{N}$. We claim that $\mathbb{P}\text{-}\lim_{n\to\infty} \phi^n_{\tau_\ell} = 1$. To see this, first observe that $\mathbb{P}\text{-}\liminf_{n\to\infty} \phi^n_{\tau_\ell} \geq 1$ holds, in the sense that $\liminf_{n\to\infty} \mathbb{P}[\phi^n_{\tau_\ell} > 1-\epsilon] \geq \liminf_{n\to\infty} \mathbb{P}[{}^\alpha\widetilde{X}^n_{\tau_\ell}/{}^\alpha\widehat{X}^n_{\tau_\ell} > 1-\epsilon] = 1$ holds for all $\epsilon \in (0,1)$. Then, given that $\mathbb{P}\text{-}\liminf_{n\to\infty} \phi^n_{\tau_\ell} \geq 1$, if $\limsup_{n\to\infty} \mathbb{P}\,[\phi^n_{\tau_\ell} > 1+\epsilon] > 0$ was true, one would conclude that $\limsup_{n\to\infty} \mathbb{E}[\phi^n_{\tau_\ell}] > 1$, which contradicts the fact that $\mathbb{E}$ $phi^n_{\tau_\ell}] \leq \phi^n_0 = 1$ holds for all $n \in \mathbb{N}$. Therefore, $\limsup_{n\to\infty} \mathbb{P}[\phi^n_{\tau_\ell} > 1+\epsilon\,] = 0$ holds for all $\epsilon \in (0,1)$, which combined with $\mathbb{P}\text{-}\liminf_{n\to\infty} \phi^n_{\tau_\ell} \geq 1$ gives $\mathbb{P}\text{-}\lim_{n\to\infty} \phi^n_{\tau_\ell} = 1$. To recapitulate, the setting is the following: $(\phi^n)_{n\in\mathbb{N}}$ is a sequence of nonnegative local martingales with $\phi^n_0 = 1$, and $\mathbb{P}\text{-}\lim_{n\to\infty} \phi^n_{\tau_\ell} = 1$ holds. In that case, lemma 2.11 in Kardaras (2013) implies that $\mathbb{P}\text{-}\lim_{n\to\infty}(\phi^n)^*_{\tau_\ell} = 1$ holds as well. Note that $(\phi^n)^* = ((\chi^n)^*)^{1-\alpha}$, so that $\mathbb{P}\text{-}\lim_{n\to\infty}(\chi^n)^*_{\tau_\ell} = 1$ holds as well. Then, the bounds in (A.8) imply that $\mathbb{P}\text{-}\lim_{n\to\infty} \chi^n_{\tau_\ell} = 1$.

Once again, we are in the following setting: $(\chi^n)_{n\in\mathbb{N}}$ is a sequence of nonnegative local martingales with $\chi^n_0 = 1$, and $\mathbb{P}\text{-}\lim_{n\to\infty} \chi^n_{\tau_\ell} = 1$ holds. An application of proposition 2.7 and lemma 2.12 in Kardaras (2013) gives that $\mathcal{S}_{\tau_\ell}\text{-}\lim_{n\to\infty} \chi^n = 1$, which also implies that $\mathcal{S}_{\tau_\ell}\text{-}\lim_{n\to\infty} \widetilde{X}^n = \widehat{X}$ by proposition 2.10 in Kardaras (2013). This implies that $\lim_{n\to\infty} \mathbb{P}[\sup_{t\in[0,\tau_\ell]} |(\widetilde{X}^n)^*_t - \widehat{X}^*_t| > \epsilon] = 0$ also holds for all $\epsilon > 0$ by Remark 4.6. Therefore, by (2.3) and lemma 2.9 in Kardaras (2013), we obtain that $\mathcal{S}_{\tau_\ell}\text{-}\lim_{n\to\infty} {}^\alpha\widetilde{X}^n = {}^\alpha\widehat{X}$, which completes the proof of Theorem 4.7.

APPENDIX B: A CAUTIONARY NOTE REGARDING THEOREM 4.7

In this section, we elaborate on the point that is made in Remark 4.9 via use of an example. In the discussion that follows, fix $\alpha \in (0,1)$. The model is the general one described in Subsection 2.1, and Assumption 2.2 is always in force.

Let $T_{1/2} = 0$ and, using induction, for $n \in \mathbb{N}$ define

$$T_n := \inf\left\{t \in \left(T_{n-1/2}, \infty\right) \mid \widehat{X}_t = \alpha\widehat{X}^*_t\right\}, \quad T_{n+1/2} := \inf\left\{t \in (T_n, \infty) \mid \widehat{X}_t = \widehat{X}^*_{T_n}\right\}.$$

(The stopping times T and τ defined in the proof of Proposition 3.13 are exactly the stopping times T_1 and $T_{3/2}$ defined above.) Note the following: $T_{n-1/2}$ is a time of maximum of $\widehat{X}$ for all $n \in \mathbb{N}$, $(T_{k/2})_{k\in\mathbb{N}}$ is an increasing sequence, and $\mathbb{P}\,[\lim_{n\to\infty} T_n = \infty] = 1$ holds. Under Assumption 2.2, Lemma A.3 implies that $\mathbb{P}\,[T_n < \infty] = 1$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, one can explicitly describe the wealth process ${}^\alpha\widetilde{X}^n$ that has the numéraire property in the class ${}^\alpha\mathcal{X}$ for investment over the interval $[0, T_n]$. In words, ${}^\alpha\widetilde{X}^n$ will follow ${}^\alpha\widehat{X}$ until time $T_{n-1/2}$, then switch to investing like the numéraire portfolio $\widehat{X}$ up to time T_n and, since at time T_n one hits the hard drawdown constraint, ${}^\alpha\widetilde{X}^n$ will remain constant from T_n onward. In mathematical terms, define

$$\begin{aligned}
{}^\alpha\widetilde{X}^n &:= {}^\alpha\widehat{X}\,\mathbb{I}_{[0,T_{n-1/2}[} + \left(\frac{{}^\alpha\widehat{X}_{T_{n-1/2}}}{\widehat{X}_{T_{n-1/2}}}\right)\widehat{X}\,\mathbb{I}_{[T_{n-1/2},T_n[} + \left(\frac{{}^\alpha\widehat{X}_{T_{n-1/2}}}{\widehat{X}_{T_{n-1/2}}}\right)\widehat{X}_{T_n}\mathbb{I}_{[T_n,\infty[} \\
&= {}^\alpha\widehat{X}\,\mathbb{I}_{[0,T_{n-1/2}[} + \left(\widehat{X}_{T_{n-1/2}}\right)^{-\alpha}\widehat{X}\,\mathbb{I}_{[T_{n-1/2},T_n[} + \left(\widehat{X}_{T_{n-1/2}}\right)^{-\alpha}\alpha\,\widehat{X}^*_{T_n}\mathbb{I}_{[T_n,\infty[},
\end{aligned}$$

where for the equality in the second line the facts that ${}^\alpha\widehat{X}_{T_{n-1/2}} = (\widehat{X}_{T_{n-1/2}})^{1-\alpha}$ and $\widehat{X}_{T_n} = \alpha\,\widehat{X}^*_{T_n}$ were used. It is straightforward to check that ${}^\alpha\widetilde{X} \in {}^\alpha\mathcal{X}$, in view of the definition of the stopping times $(T_{k/2})_{k\in\mathbb{N}}$. Pick any $X \in \mathcal{X}$. The global (in time) numéraire property of $\widehat{X}$ in $\mathcal{X}$ will give

$$\mathbb{E}\left[\frac{{}^{\alpha}X_{T_n}}{{}^{\alpha}\widetilde{X}^n_{T_n}} - 1 \,\middle|\, \mathcal{F}_{T_{n-1/2}}\right] \le \frac{{}^{\alpha}X_{T_{n-1/2}}}{{}^{\alpha}\widetilde{X}^n_{T_{n-1/2}}} - 1 = \frac{{}^{\alpha}X_{T_{n-1/2}}}{{}^{\alpha}\widehat{X}^n_{T_{n-1/2}}} - 1.$$

Upon taking expectation on both sides of the previous inequality, we obtain $\mathbb{E}\mathrm{rr}_{T_n}({}^{\alpha}X|{}^{\alpha}\widetilde{X}^n) \le \mathbb{E}\mathrm{rr}_{T_{n-1/2}}({}^{\alpha}X|{}^{\alpha}\widehat{X}^n) \le 0$, the last inequality holding in view of statement (2) of Theorem 3.8, given that $T_{n-1/2}$ is a time of maximum of $\widehat{X}$. We have shown that ${}^{\alpha}\widetilde{X}^n$ indeed has the numéraire property in the class ${}^{\alpha}\mathcal{X}$ for investment over the interval $[0, T_n]$.

Note that ${}^{\alpha}\widetilde{X}^n = {}^{\alpha}\widehat{X}$ identically holds in the stochastic interval $[0, T_{n-1/2}]$ for each $n \in \mathbb{N}$; therefore, the conclusion of Theorem 4.7 in this case is valid in a quite strong sense. However, the behavior of ${}^{\alpha}\widetilde{X}^n$ and ${}^{\alpha}\widehat{X}$ in the stochastic interval $[T_{n-1/2}, T_n]$ is different and results in quite diverse outcomes at time T_n, as we shall now show. At time T_n one has

$${}^{\alpha}\widehat{X}_{T_n} = \alpha\left(\widehat{X}^*_{T_n}\right)^{1-\alpha} + (1-\alpha)\left(\widehat{X}^*_{T_n}\right)^{-\alpha}\widehat{X}_{T_n} = \alpha(2-\alpha)\left(\widehat{X}^*_{T_n}\right)^{1-\alpha},$$

where the fact that $\widehat{X}_{T_n} = \alpha\widehat{X}^*_{T_n}$ was again used. Furthermore, ${}^{\alpha}\widetilde{X}^n_{T_n} = (\widehat{X}_{T_{n-1/2}})^{-\alpha}\alpha\widehat{X}^*_{T_n}$. It then follows that

$$\frac{{}^{\alpha}\widetilde{X}^n_{T_n}}{{}^{\alpha}\widehat{X}_{T_n}} = \frac{\left(\widehat{X}_{T_{n-1/2}}\right)^{-\alpha}\alpha\widehat{X}^*_{T_n}}{\alpha(2-\alpha)\left(\widehat{X}^*_{T_n}\right)^{1-\alpha}} = \frac{1}{2-\alpha}\left(\frac{\widehat{X}^*_{T_n}}{\widehat{X}_{T_{n-1/2}}}\right)^{\alpha} =: \zeta_n.$$

In view of Assumption 2.2 and the result of Dambis, Dubins, and Schwarz—see theorem 3.4.6 in Karatzas and Shreve (1991)—the law of the random variable ζ_n is the same for all $n \in \mathbb{N}$. In fact, universal distributional properties of the maximum of a nonnegative local martingale stopped at first hitting time—see proposition 4.3 in Carraro et al. (2012)—imply that $\zeta_n = (2-\alpha)^{-1}(\alpha + (1-\alpha)(1/\eta_n))^{\alpha}$, where η_n has the uniform law on $(0, 1)$. In particular, $\mathbb{P}[\zeta_n < (2-\alpha)^{-1} + \epsilon] > 0$ and $\mathbb{P}[\zeta_n > (2-\alpha)^{-1} + \epsilon^{-1}] > 0$ holds for all $\epsilon \in (0, 1)$. Furthermore, ζ_n is $\mathcal{F}_{T_n}$-measurable and independent of $\mathcal{F}_{T_{n-1/2}} \supseteq \mathcal{F}_{T_{n-1}}$ for each $n \in \mathbb{N}$, which implies that $(\zeta_n)_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables. By an application of the second Borel–Cantelli lemma, it follows that

$$\frac{1}{2-\alpha} = \liminf_{n\to\infty}\left(\frac{{}^{\alpha}\widetilde{X}^n_{T_n}}{{}^{\alpha}\widehat{X}_{T_n}}\right) < \limsup_{n\to\infty}\left(\frac{{}^{\alpha}\widetilde{X}^n_{T_n}}{{}^{\alpha}\widehat{X}_{T_n}}\right) = \infty,$$

demonstrating the claim made at Remark 4.9.

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