Speculative Trading of Electricity Contracts in Interconnected Locations☆

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Abstract

We derive an investor's optimal trading strategy of electricity contracts traded in two locations joined by an interconnector. The investor employs a price model which includes the impact of her own trades. The investor's trades have a permanent impact on prices because her trading activity affects the demand of contracts in both locations. Additionally, the investor receives prices which are worse than the quoted prices as a result of the elasticity of liquidity provision of contracts. Furthermore, the investor is ambiguity averse, so she acknowledges that her model of prices may be misspecified and considers other models when devising her trading strategy. We show that as the investor's degree of ambiguity aversion increases, her trading activity decreases in both locations, and thus her inventory exposure also decreases. Finally, we show that there is a range of ambiguity aversion parameters where the Sharpe ratio of the trading strategy increases when ambiguity aversion increases.

Keywords: Ambiguity aversion, model uncertainty, electricity interconnector, statistical arbitrage

1. Introduction

In this paper we show how an investor maximizes profits by taking simultaneous and offsetting positions in electricity contracts traded in two locations, which are joined by an electricity interconnector. The investor employs a model that incorporates two ways in which her trades affect the prices of contracts. When the investor takes liquidity by executing a

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trade (buy or sell contracts), the prices she receives are worse than the prevailing quoted prices as a result of the inelasticity of the liquidity of the supply of contracts. Moreover, the buy and sell pressure stemming from the investor's trades have a permanent effect on the prices of the contracts in both locations.

The investor acknowledges that her model of prices, which is characterized by the reference measure $\mathbb{P}$, may be misspecified. She deals with this model uncertainty, also referred to as ambiguity aversion, by considering alternative models when developing the optimal trading strategy. Specifically, the investor considers models characterized by probability measures that are absolutely continuous with respect to the reference model $\mathbb{P}$. The decision to reject the reference measure is based on a penalty that the investor incurs if she adopts an alternative model. The magnitude of the penalty depends on the investor's degree of ambiguity aversion and is based on a 'measure' of the distance between the reference and the alternative measure.

In this paper we solve the investor's optimal trading problem and show how ambiguity aversion affects the trading strategy. We find that as the investor's degree of ambiguity aversion increases, her trading activity in both markets decreases, hence her inventory holdings (long or short) decrease, and this has an effect on the strategy's expected profits and volatility of profits. In particular, we employ simulations to illustrate the behavior of the strategy and measure its financial performance by computing the ratio of the average profits to the standard deviation of profits (i.e., Sharpe ratio assuming zero risk-free rate). We show that there is a range of ambiguity aversion parameters where the Sharpe ratio increases as ambiguity aversion increases.

The recent work of Cartea et al. (2016) is the first to study the effect of model uncertainty in commodities. The authors showed how the prices that consumers are willing to pay, and producers to receive, for forward contracts and other derivatives, depend on the degree of confidence that they place on their reference measure. Model uncertainty has been used in several other settings in the literature. For applications in portfolio optimization and consumption problems, see for instance Hansen and Sargent (2001), Uppal and Wang (2003), Hansen and Sargent (2007), and Guidolin and Rinaldi (2013); in credit derivatives Jaimungal and Sigloch (2012); in algorithmic trading Cartea et al. (2014); in real-options Cartea and Jaimungal (2015).

Another line of research is that of Bannor et al. (2013), where the authors investigate parameter uncertainty in energy markets. Our approach is different because the investor deals with model uncertainty by considering a large class of alternative models. This class consists of all models described by a probability measure, where the only requirement is that the measures are equivalent to the investor's model which is characterized by the reference measure $\mathbb{P}$.

Previous work on electricity interconnectors include Cartea and González-Pedraz (2012) who develop a tool to value an interconnector as a stream of options written on the spread of electricity spot prices in two locations. Our paper differs from that by Cartea and González-Pedraz (CG) in three important aspects. First, in our approach the investor trades electricity contracts instead of spot electricity. Second, our investor solves a dynamic optimization problem, whereas CG solve a static problem that is not dynamically optimal. Third, we model the investor's price impact, whereas CG incorporate price impact by assuming that the value of the option to transmit electricity through the interconnector is capped at an arbitrary level.

In this paper the investor employs a reference model that is in the class of arithmetic models as in Benth et al. (2007). However, there is a large list of reduced-form models for wholesale power prices and electricity derivatives. For example, Benth et al. (2003) model electricity forward prices, while Roncoroni (2002), Cartea and Figueroa (2005), Weron (2007), propose models for wholesale electricity prices, see also Benth and Saltyte-Benth (2006), Hikspoors and Jaimungal (2007), Benth et al. (2008), and Jaimungal and Surkov (2011). Moreover, Hambly et al. (2009) model power prices and price different types of energy derivatives, including swing options, designed to manage the risk exposure of producers and consumers. Finally, Aïd (2015) provides a survey of the common features of the microstructure of power markets and models, and describes the state of the art of the different electricity price models to price derivatives.

The remainder of this paper is organized as follows. Section 2 presents the investor's reference model for the price dynamics of electricity contracts. Section 3 shows how model uncertainty affects the investor's trading strategy. Section 4 derives the investor's optimal trading between the two interconnected locations. Section 5 employs simulations to illustrate the trading strategy and shows the financial performance of the strategy. Section 6 concludes and proofs are collected in Section 7.

2. The Model

In this section we develop the framework for an investor who trades electricity contracts in two locations joined by an electricity interconnector. To streamline the discussion, Subsection 2.1 presents the investor's reference model for the dynamics of the midprices of the contracts in the absence of her own trading activity. Subsection 2.2 shows the prices received by the investor and how her trading activity affects supply and demand of the electricity contracts in both locations.

2.1. Midprice dynamics

Throughout this paper we focus on one type of contract and assume it is traded in both locations. The midprice of the contract is denoted by $P^i = (P^i_t)_{t\in[0,T]}$, and, in the absence of the investor's trading activity, the investor assumes the following model for the midprice dynamics:

$$dP^i_t = \kappa^i(\theta^i_t - P^i_t)\,dt + \sigma^i\,dW^i_t + \,dJ^i_t\,, \tag{1}$$

where $i \in \{1, 2\}$ denotes location, $\kappa^i \geq 0$, $\sigma^i \geq 0$ are constants, $J^i = (J^i_t)_{\{0\leq t\leq T\}}$ are compensated pure jump process, $W^i = (W^i_t)_{t\in[0,T]}$ are standard $\mathbb{P}$-Brownian motions with correlation $d\,[\,W^1,\,W^2\,]_t = \rho\,dt$, $\rho \in [1, -1]$, and the function $\theta^i = (\theta^i_t)_{t\in[0,T]}$ represents the deterministic seasonal component of prices. The jumps within and across locations are independent, and the jumps and the Brownian motions are independent of each other. As usual, we work on a completed filtered probability space $(\Omega,\,\{\mathcal{F}_t\}_{t\in[0,T]},\,\mathbb{P})$, where $\mathbb{P}$ is a probability measure, and $\mathcal{F}_t$ is the natural filtration generated by the quadruplet $(W^1,\,W^2,\,J^1,\,J^2)$.

To generate the alternative models in the sequel, it is useful to first write the jump process J^i in terms of its Poisson random measure (PRM) $\mu_i(dy, dt)$, and compensator $\nu_i(dy, dt)$, so that

$$J^i_t = \int_0^t \int_{-\infty}^{\infty} y\;[\,\mu_i(dy, du) - \nu_i(dy, du)\,]\;.$$

Furthermore, we assume that the compensator can be written in the form $\nu_i(dy, du) = \lambda_i\,G_i(dy)\,du$ so that the pure jump process J^i is a pure jump Lévy process. Moreover, if $\int_{-\infty}^{\infty} G_i(dy) = 1$, J^i is a homogeneous compound Poisson process with intensity λ_i.

2.2. Price impact of investor's trading across locations

The investor's objective is to maximize expected profits by taking advantage of price discrepancies in both locations. We assume that the investor is continuously trading and takes simultaneous and offsetting positions in both locations. The speed at which she trades is denoted by $\nu = (\nu_t)_{t\in[0,T]}$ and has the following interpretation: over a small time step Δt, if $\nu_t > 0$, the investor (i) buys $\nu_t\,\Delta t$ of the contract in location 1, and (ii) sells the same quantity of contracts in location 2. Similarly, if $\nu_t < 0$ the investor sells $\nu_t\,\Delta t$ units of electricity in location 1 and buys the same amount in location 2.

The investor's trading activity affects the prices of the contracts in each location because her trading increases the quantity demanded (to buy or sell) of electricity contracts in both markets. We label this effect: permanent price impact. In addition, when the investor executes orders, the price she receives may be worse than the quoted marginal prices to buy and sell contracts. We label this effect: temporary price impact. We discuss both effects in more detail and show how they are modelled in our setup.

Permanent price impact

When the investor buys electricity contracts in location i, the demand in that location increases and exerts an upward pressure on clearing prices. Similarly, when the investor sells electricity contracts in the other location, she exerts a downward pressure on clearing prices. The impact on midprices depends on the magnitude of the buying and selling pressure, which results from the investor's rate of trading. We assume that the impact on midprices is permanent and linear in the investor's speed of trading. Specifically, if over the infinitesimal time-step dt the investor purchases an amount $\nu_t\, dt$ of electricity contracts in location $i = 1$, then the price in that location will drift upwards by $b_1\, \nu_t\, dt$, and downward in location 2 by $b_2\, \nu_t\, dt$, where $b_i \geq 0$ is the permanent price impact parameter. Note that if the investor is selling contracts in location 1 then $\nu_t < 0$, so midprices are pushed down in location 1 and up in location 2.

Therefore, when the investor trades in both locations her reference model is as in (1), but with the additional price impact components. Thus, the midprice dynamics of the electricity contracts become

$$dP_t^{1,\nu} = \kappa_1\,(\theta_t^1 - P_t^{1,\nu})\,dt + b_1\,\nu_t\,dt + \sigma_1\,dW_t^1 + dJ_t^1\,, \tag{2a}$$
$$dP_t^{2,\nu} = \kappa_2\,(\theta_t^2 - P_t^{2,\nu})\,dt - b_2\,\nu_t\,dt + \sigma_2\,dW_t^2 + dJ_t^2\,, \tag{2b}$$

where we use the notation $P^{i,\nu}$ to stress that midprices are affected by the investor's (controlled) speed of trading, and recall that all the parameters and sources of risk are under the reference measure $\mathbb{P}$.

Temporary price impact

When the investor trades she receives prices that are worse than the prevailing midprice she observes. This difference results from the immediate buy and sell pressure exerted by the investor's trading activity and the elasticity of the liquidity of contracts supplied in both locations. For example, if the investor is selling electricity contracts and the liquidity offered by counterparties is very inelastic, then the price she receives will be lower than the price at which the market would have cleared, ceteris paribus, in the absence of the investor's sell pressure.

We denote by $\hat{P}_t^{i,\nu}$ the execution prices the investor receives:

$$\hat{P}_t^{1,\nu} = P_t^{1,\nu} + a_1\,\nu_t\,, \tag{3a}$$
$$\hat{P}_t^{2,\nu} = P_t^{2,\nu} - a_2\,\nu_t\,, \tag{3b}$$

where $P_t^{i,\nu}$ are the quoted midprices as in (2), and $a_i \geq 0$ are the temporary price impact parameters.

3. Trading in interconnected locations

So far we have described the model under the measure $\mathbb{P}$, which is the investor's view on market dynamics for the contracts she trades. However, the investor acknowledges that her model may be misspecified, so she is willing to entertain other models to devise an optimal trading strategy. In this section we show how the investor considers alternative models to the reference measure $\mathbb{P}$, and how she chooses a particular model from all available choices.

3.1. Model uncertainty

The investor is not confident about the model for price dynamics, so she considers other models specified by a measure $\mathbb{Q}$ which is equivalent to the reference measure $\mathbb{P}$. We denote by $\mathcal{Q}$ the set of equivalent measures that the agent considers – below we provide the mathematical details of this set.

Furthermore, the investor requires a criteria to select a model in the set $\mathcal{Q}$, which trivially contains $\mathbb{P}$. As part of the selection process the investor assumes that she will incur a cost if the reference model is rejected in favor of an alternative model. Intuitively, the rejection cost is based on a measure of 'distance' between the reference measure $\mathbb{P}$ and the alternative model $\mathbb{Q}$. This measure of distance is encoded in a penalty function which also accounts for the investor's degree of ambiguity aversion. For instance, if the investor is very confident about the reference model, any 'small' deviation, i.e. small distance, from the reference measure $\mathbb{P}$ is heavily penalized, so it is very costly to reject the reference measure. On the other hand, if the investor is extremely ambiguous (i.e., very underconfident) about her choice of the reference measure, the cost incurred for accepting alternative models is very small.

Before providing details of the penalty function, we specify the investor's inventory and cash process to formalize the investment problem. Let $Q^\nu = (Q_t^\nu)_{\{0\leq t\leq T\}}$ denote the inventory in electricity contracts in location 1. This inventory is affected by how fast she trades and satisfies

$$dQ_t^\nu = \nu_t\, dt\,, \qquad Q_0^\nu = 0\,. \tag{4}$$

Recall that the investor takes offsetting positions in both markets, so the inventory in location 2 is $-Q_t^\nu$.

The cash accumulated from trading is given by the process $X^\nu = (X_t^\nu)_{\{0\leq t\leq T\}}$ and satisfies

$$X_t^\nu = \int_0^t f(u, \boldsymbol{P}_u, \nu_u)\, \nu_u\, du\,, \qquad X_0^\nu = 0\,, \tag{5}$$

where $\boldsymbol{P} = (P_t^{1,\nu}, P_t^{2,\nu})_{t\in[0,T]}$, $f(t, \boldsymbol{P}_t, \nu_t) = P_t^{2,\nu} - P_t^{1,\nu} - a\,\nu_t$ is the instantaneous profit from taking simultaneous positions in both locations, and $a = a_1 + a_2$ denotes the aggregate temporary price impact in both locations.

The investor's value function is (with a slight abuse of notation $\boldsymbol{P}$ represents the point in state space corresponding to the midprices)

$$H(t, \boldsymbol{P}, q) = \sup_{\nu \in \mathcal{A}} \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}_{t,\boldsymbol{P},q} \left[\int_t^T f(u, \boldsymbol{P}_u, \nu_u)\, du \, + g(T, \boldsymbol{P}_T, q_T) + \mathcal{H}(\mathbb{Q}|\mathbb{P}) \,\middle|\, \mathcal{F}_t \right] , \tag{6}$$

where

$$g(T, \boldsymbol{P}_T, q_T) = (P^1_T - P^2_T)\, q_T - \alpha\, q_T^2 \,, \tag{7}$$

is the cost of unwinding all contracts at the terminal date T, and the constant $\alpha \geq 0$ represents liquidation costs. Moreover, $\mathcal{A}$ is set of admissible trading strategy $\mathcal{A} = \{\nu \, : \, \mathbb{E}^{\mathbb{P}}[\int_0^T \nu_u^2\, du] < +\infty\}$, and the operator $\mathbb{E}^{\mathbb{Q}}_{t,\boldsymbol{P},q}[\,\cdot\,]$ denotes expectation conditioned on (with slight abuse of notation) $P^1_{t^-} = P^1$, $P^2_{t^-} = P^2$, and $Q_t = q$. Furthermore, $\mathcal{H}(\mathbb{Q}|\mathbb{P}) \geq 0$ is the (convex) penalty function, i.e. the cost of choosing a candidate measure $\mathbb{Q}$ over the reference model $\mathbb{P}$.

Now we discuss the penalty imposed by the investor when adopting an alternative measure. We build this penalty in three steps. First, we discuss the penalty imposed when deviating from the reference measure for only the diffusive factor. Second, we discuss the penalty imposed when deviating from the reference measure for only the jump factor. Finally, we show how these two penalties are combined to obtain the general penalty $\mathcal{H}(\mathbb{Q}|\mathbb{P})$ that appears in the value function (6).

3.2. Penalty function: cost of adopting an alternative model

A popular choice for the penalty function is based on relative entropy:

$$\widehat{\mathcal{H}}_{t,T}(\mathbb{Q}|\mathbb{P}) = \frac{1}{\gamma} \log \frac{d\mathbb{Q}}{d\mathbb{P}} \,, \tag{8}$$

where $\gamma > 0$ is a constant which reflects the investor's degree of confidence in the reference model. In the limiting case $\gamma \downarrow 0$, the investor is ambiguity neutral and therefore rejects any alternative model because the cost of adopting an alternative measure is too high. On the other hand, the more ambiguous is the investor about the reference model the larger is γ. In the extreme case $\gamma \to \infty$ the investor contemplates the worst case scenario.

Now we define the class of alternative models that the investor considers. The investor is ambiguous to the two sources of uncertainty in the reference model $\mathbb{P}$: the diffusive factor and the jump factor. We characterize the class of equivalent measures for each factor separately. The investor may feel more or less confident about the reference measure for the diffusion than for the jump component – Bannor et al. (2013) find that in wholesale electricity markets, jump risk is by far the most important source of model risk, see also Stahl et al. (2012). In addition, the investor could also feel more or less ambiguous to the reference measure for each location.

Ambiguity aversion to diffusive factors. The investor considers alternative models to the diffusive factors characterized by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^{\boldsymbol{\eta}}}{d\mathbb{P}} = \exp\left\{-\frac{1}{2}\int_0^T \boldsymbol{\eta}_u' \, \boldsymbol{\Sigma}^{-1} \, \boldsymbol{\eta}_u \, du - \int_0^T \boldsymbol{\eta}_u' \, d\boldsymbol{W}_u\right\} , \tag{9}$$

where $\boldsymbol{W}_t = (W_t^1, W_t^2)'$, $\boldsymbol{\eta} = (\boldsymbol{\eta}_t)_{t\in[0,T]}$ is a two-dimensional $\mathcal{F}_t$-adapted process, the matrix transpose operator is denoted by $'$, and

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} , \qquad \text{so that} \qquad \boldsymbol{W}_t^{\eta} = -\int_0^t \boldsymbol{\eta}_u \, du + \boldsymbol{W}_t$$

are $\mathbb{Q}^{\boldsymbol{\eta}}$-standard Brownian motions. This change of measure is parameterized by the $\mathcal{F}$-predictable process $\boldsymbol{\eta}$, which changes the drift of the reference model. In addition, the set of candidate measures

$$\mathcal{Q}^{\eta} = \left\{\mathbb{Q}^{\eta}(\eta) \; : \; \boldsymbol{\eta} \text{ is } \mathcal{F}\text{-predictable and } \mathbb{E}^{\mathbb{P}}\left[\int_0^T \boldsymbol{\eta}_u' \, \boldsymbol{\Sigma}^{-1} \, \boldsymbol{\eta}_u \, du\right] < \infty\right\} , \tag{10}$$

and the entropic penalization specific to the diffusive factor in the model is as in (8), and here takes the form

$$\mathcal{H}^{\boldsymbol{\Phi}}(\mathbb{Q}^{\eta}|\mathbb{P}) = -\frac{1}{2}\int_0^T \boldsymbol{\eta}_u' \, \boldsymbol{\Sigma}^{-1} \, \boldsymbol{\eta}_u \, du - \int_0^T \boldsymbol{\eta}_u' \, d\boldsymbol{W}_u \, ,$$

where $\boldsymbol{\Phi}$ is an ambiguity matrix with inverse given by

$$\boldsymbol{\Phi}^{-1} = \phi \, \boldsymbol{\Sigma}^{-1} + \phi_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \phi_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} . \tag{11}$$

Here $\phi \geq 0$ is an ambiguity aversion parameter common to the diffusion component of the model for midprices P^1 and P^2 in both locations, and $\phi_1 \geq 0$ and $\phi_2 \geq 0$ are ambiguity aversion parameters for the midprices P^1 and P^2 respectively.

Finally, the $\mathbb{Q}^{\boldsymbol{\eta}}$-expectation of the penalty function, specific to the diffusive factor, is

$$\mathbb{E}^{\mathbb{Q}^{\boldsymbol{\eta}}}\left[\mathcal{H}^{\boldsymbol{\Phi}}(\mathbb{Q}^{\eta}|\mathbb{P})\right] = \mathbb{E}^{\mathbb{Q}^{\boldsymbol{\eta}}}\left[\frac{1}{2}\int_0^T \boldsymbol{\eta}_u' \, \boldsymbol{\Phi}^{-1} \, \boldsymbol{\eta}_u \, du\right] .$$

Ambiguity aversion to jump factor. Analogously, the alternative models to the jump factor are parameterized by the $\mathcal{F}$-predictable random field $\boldsymbol{g} = (\boldsymbol{g}_t(\cdot))_{t\in[0,T]}$, and characterized by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^{g^i}}{d\mathbb{P}} = \exp\left\{-\int_0^T \int_{-\infty}^{\infty} \left(e^{g_u^i(y)} - 1\right) \nu(dy, du) + \int_0^T \int_{-\infty}^{\infty} g_u^i(y) \, \mu(dy, du)\right\} , \tag{12}$$

where $\boldsymbol{g}_t := (g_t^1, g_t^2)$.

$$
\begin{tikzcd}
& \mathbb{Q}^{\eta} \arrow[dr, "\varepsilon, g_t(y)"] & \\
\mathbb{P} \arrow[ur, "\boldsymbol{\Phi}, \boldsymbol{\eta}_t"] \arrow[rr, dotted] \arrow[dr, "\varepsilon, g_t(y)"'] & & \mathbb{Q}^{\boldsymbol{\eta},\boldsymbol{g}} \\
& \mathbb{Q}^{g} \arrow[ur, "\boldsymbol{\Phi}, \boldsymbol{\eta}_t"'] &
\end{tikzcd}
$$

Figure 1: Two alternative routes from the reference measure $\mathbb{P}$ to a candidate measure $\mathbb{Q}^{\boldsymbol{\eta},\boldsymbol{g}}$ in which the diffusion and jump component are altered.

The $\mathbb{Q}^{g^i}$-compensator of $\mu_i(dy, dt)$ is then

$$
\nu_{\mathbb{Q}^{g^i}}(dy, dt) = e^{g_t^i(y)} \nu_i(dy, dt)\,, \tag{13}
$$

and we choose the class of candidate measures

$$
\mathcal{Q}^{g^i} = \left\{ \mathbb{Q}^{g^i} \;:\; g^i(\cdot) \text{ is } \mathcal{F}\text{-predictable and } \mathbb{E}^{\mathbb{P}}\left[\int_0^T \int_{-\infty}^{\infty} (g_u^i(y))^2 \, \nu_i(dy, du) \right] < \infty \right\}, \tag{14}
$$

and penalty function

$$
\mathcal{H}^{\varepsilon}(\mathbb{Q}^{g^i}|\mathbb{P}) = \frac{1}{\varepsilon}\left(-\int_0^T \int_{-\infty}^{\infty} \left(e^{g_u^i(y)} - 1 \right) \nu_i(dy, du) + \int_0^T \int_{-\infty}^{\infty} g_u^i(y) \, \mu_i(dy, du) \right),
$$

where $\varepsilon \geq 0$ is the ambiguity aversion parameter specific to the jump factor, and the $\mathbb{Q}^g$-expectation of the penalty is

$$
\mathbb{E}^{\mathbb{Q}^{g^i}}\left[\mathcal{H}^{\varepsilon}(\mathbb{Q}^{g^i}|\mathbb{P}) \right] = \mathbb{E}^{\mathbb{Q}^{g^i}}\left[\frac{1}{\varepsilon} \int_0^T \int_{-\infty}^{\infty} \left(e^{g_u^i(y)}(g_u^i(y) - 1) + 1 \right) \nu_i(dy, du) \right].
$$

3.2.1. Ambiguity aversion to diffusive and jump factors

Now that we have specified the set of candidate models for each individual factor, we discuss the set of candidate measures that the investor considers when accounting for ambiguity to both factors at the same time. Thus we seek a 'total' change of measure such that $\mathbb{P} \xrightarrow{\boldsymbol{\Phi}, \boldsymbol{\varepsilon}} \mathbb{Q}^{\boldsymbol{\eta},\boldsymbol{g}}$.

Since the PRM driving the jumps in midprices and the OU process are mutually independent, a Radon-Nikodym derivative that generates an equivalent measure can be factored into the product of two independent components. Figure 1 shows that this decomposition can be viewed as two consecutive measure changes which yield the desired total measure change. This total measure change can be reached via two canonical paths: (i) alter only the diffusive components, and then alter the jump components, or (ii) alter the jump components, and then alter the diffusive components, that is,

$$
\frac{d\mathbb{Q}^{\boldsymbol{\eta},\boldsymbol{g}}}{d\mathbb{P}} = \frac{d\mathbb{Q}^{\boldsymbol{\eta}}}{d\mathbb{P}} \cdot \frac{d\mathbb{Q}^{\boldsymbol{\eta},\boldsymbol{g}}}{d\mathbb{Q}^{\boldsymbol{\eta}}} \qquad \text{or} \qquad \frac{d\mathbb{Q}^{\boldsymbol{\eta},\boldsymbol{g}}}{d\mathbb{P}} = \frac{d\mathbb{Q}^{g}}{d\mathbb{P}} \cdot \frac{d\mathbb{Q}^{\boldsymbol{\eta},\boldsymbol{g}}}{d\mathbb{Q}^{g}}\,.
$$

Both paths arrive at the same total change of measure, which is given by the Radon-Nikodym derivative

$$\begin{aligned}\frac{d\mathbb{Q}^{\boldsymbol{\eta},\boldsymbol{g}}}{d\mathbb{P}} = \exp\Bigg\{ & -\tfrac{1}{2}\int_0^T \boldsymbol{\eta}_u'\,\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\eta}_u\,du - \int_0^T \boldsymbol{\eta}_u'\,d\boldsymbol{W}_u \\ & - \sum_{i=1,2}\left[\int_0^T\int_{-\infty}^{\infty}\left(e^{g_u^i(y)}-1\right)\,\nu_i(dy,du) - \int_0^T\int_{-\infty}^{\infty} g_u^i(y)\,\mu_i(dy,du)\right]\Bigg\}.\end{aligned} \tag{15}$$

We remark that while this measure change is induced by a product of stochastic exponentials, the resulting probability measure may induce dependence between the various processes since the random fields $\boldsymbol{g}_t(\cdot)$ and the processes $\boldsymbol{\eta}_t$ are allowed to be dependent on all state variables.

Finally, the set of candidate measures is

$$\begin{aligned}\mathcal{Q} = \Bigg\{ & \mathbb{Q}^{\boldsymbol{\eta},\boldsymbol{g}} : \boldsymbol{\eta} \text{ and } \boldsymbol{g} \text{ are } \mathcal{F}_t - \text{predictable}, \ \mathbb{E}^{\mathbb{P}}\left[\int_0^T \boldsymbol{\eta}_u'\,\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\eta}_u\,du\right] < \infty\,, \\ & \mathbb{E}^{\mathbb{P}}\left[\sum_{i=1,2}\int_0^T\int_{-\infty}^{\infty}(g_u^i(y))^2\,\nu_i(dy,du)\right] < \infty \Bigg\}.\end{aligned} \tag{16}$$

3.3. Dynamic programming equation

The dynamic programming equation to associated with the optimal control problem (6), suggests that the value function $H(t,\boldsymbol{P},q)$ is the unique solution of the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation:

$$\begin{aligned}0 = {} & \partial_t H(t,\boldsymbol{P},q) + \sup_{\nu}\left(\mathcal{L}^{\nu} + f(t,\boldsymbol{P},\nu)\right)H(t,\boldsymbol{P},q) \\ & + \sum_{i=1,2}\lambda_i \inf_{g_i}\int_{-\infty}^{\infty}\left\{e^{g_i(y)}\,\Delta_i(y)H(t,\boldsymbol{P},q) + \tfrac{1}{\varepsilon}\left(1 + e^{g_i(y)}\left(g_i(y)-1\right)\right)\right\}\,G_i(dy) \\ & + \inf_{\boldsymbol{\eta}}\left\{\boldsymbol{\eta}'\,\boldsymbol{\Omega}\,\boldsymbol{\mathcal{D}}H(t,\boldsymbol{P},q) + \tfrac{1}{2}\boldsymbol{\eta}'\,\boldsymbol{\Phi}^{-1}\,\boldsymbol{\eta}\right\}\,,\end{aligned} \tag{17}$$

where

$$\boldsymbol{\Omega} = \begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{pmatrix}, \quad \boldsymbol{\Phi}^{-1} = \phi\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}^{-1}, \quad \boldsymbol{\mathcal{D}}H = \begin{pmatrix}\partial_{P^1}H\\ \partial_{P^2}H\end{pmatrix},$$

the operators $\Delta_i(y)H$ act on functions as follows:

$$\Delta_i(y)H(t,\boldsymbol{P},q) = H(t,\boldsymbol{P}+y\mathbb{1}_i,q) - H(t,\boldsymbol{P},q)\,, \tag{18}$$

with $\mathbb{1}_1 = (1,0)$ and $\mathbb{1}_2 = (0,1)$, and the generator

$$\begin{aligned}\mathcal{L}^{\nu} = {} & \left(\kappa_1\left(\theta_1 - P^1\right) + b_1\,\nu\right)\partial_{P^1} + \left(\kappa_2\left(\theta_2 - P^2\right) - b_2\,\nu\right)\partial_{P^2} \\ & + \frac{1}{2}\sigma_1^2\partial_{P^1P^1} + \frac{1}{2}\sigma_2^2\partial_{P^2P^2} + \rho\,\sigma_1\,\sigma_2\,\partial_{P^1P^2} + \nu\,\partial_q - \sum_{i=1}^{2}\lambda_i\psi_i\partial_{P_i}\,,\end{aligned} \tag{19}$$

where

$$\psi_i = \int_{-\infty}^{\infty} y \, G_i(dy) \,,$$

is the mean jump size in location i.

For simplicity we have assumed that $\phi_1 = \phi_2 = 0$ – an assumption that we keep throughout the remainder of the paper.

Proposition 1. ***Trading under model uncertainty.***
After taking the supremum and infinum in the HJBI (17)*, we have*

$$\begin{aligned}(\partial_t + \mathcal{L})\, H(t, \boldsymbol{P}, q) &+ \tfrac{1}{4\,a} \left[(b_1 \partial_{P^1} - b_2 \partial_{P^2} + \partial_q)\, H(t, \boldsymbol{P}, q) + \left(P^2 - P^1\right)\right]^2 \\ &- \tfrac{1}{2} \left(\boldsymbol{\mathcal{D}} H(t, \boldsymbol{P}, q)\right)' \, \boldsymbol{\Omega}' \, \boldsymbol{\Phi}^{-1} \, \boldsymbol{\Omega} \left(\boldsymbol{\mathcal{D}} H(t, \boldsymbol{P}, q)\right) \\ &+ \sum_{i=1,2} \lambda_i \int_{-\infty}^{\infty} \frac{1 - e^{-\varepsilon \, \Delta_i(y) H(t, \boldsymbol{P}, q)}}{\varepsilon} \, G_i(dy) \; = \; 0 \,,\end{aligned} \tag{20}$$

subject to the terminal condition

$$H(T, \boldsymbol{P}, q) = \left(P^1 - P^2\right) \, q - \alpha \, q^2, \quad \forall P^1, P^2, q \,,$$

and the operator

$$\begin{aligned}\mathcal{L} = \kappa_1 \, &\left(\theta_1 - P^1\right) \partial_{P^1} + \kappa_2 \, \left(\theta_2 - P^2\right) \, \partial_{P^2} \\ &+ \frac{1}{2} \sigma_1^2 \, \partial_{P^1 P^1} + \frac{1}{2} \, \sigma_2^2 \, \partial_{P^2 P^2} + \rho \, \sigma_1 \, \sigma_2 \, \partial_{P^1 P^2} - \sum_{i=1}^{2} \lambda_i \psi_i \partial_{P_i} \,.\end{aligned} \tag{21}$$

Furthermore, the optimal speed of trading in feedback form is

$$\nu^*(t, \boldsymbol{P}, q) = \tfrac{1}{2\,a} \left[\, (b_1 \partial_{P^1} - b_2 \, \partial_{P^2} + \partial_q) \, H(t, \boldsymbol{P}, q) + \left(P^2 - P^1\right) \right] \,,$$

and the optimal measures for the diffusion and jump components are characterized by

$$\boldsymbol{\eta}^*(t, \boldsymbol{P}, q) = -\, \boldsymbol{\Phi} \, \boldsymbol{\Omega} \, \boldsymbol{\mathcal{D}} \, H(t, \boldsymbol{P}, q) \,, \qquad \textit{and} \qquad g_i^*(t, y, \boldsymbol{P}, q) = -\varepsilon \, \Delta_i(y) H(t, \boldsymbol{P}, q) \,, \tag{22}$$

for $i = \{1, 2\}$*, respectively.*

Proof. The optimal trading speed is obtained by maximizing the second term of (17). Since this term is quadratic in ν, it is trivial to obtain the first order condition (FOC). The coefficient of ν^2 is negative, and hence it obtains a maximum value there. Similarly, to obtain $g_i^*(\cdot)$, we observe that the maximum value of the integral in the second line of (17) is obtained by maximizing the integrand pointwise in y, and independently for each i. Furthermore, the function $\ell(z) = e^z f + \frac{1}{\varepsilon}(1+e^z(z-1))$ is convex and attains a unique minima at $z^* = -\varepsilon f$ for arbitrary $f \in \mathbb{R}$, and $\varepsilon > 0$. Next, since $\mathbf{\Phi}$ is positive semi-definite, the infimum in the third line is attained at the FOC of the quadratic form, which is given by (22). Finally, substituting the feedback forms of ν^*, η^*, and $g_i^*(\cdot)$ into (17) results in (20). ∎

4. Optimal trading across locations

The HJBI satisfied by the value function is nonlinear and for the general case when the investor is ambiguity averse to both the jump and diffusive components of the reference model, we cannot find a closed-form solution to (20). However, if the investor is ambiguity averse to only the diffusive factor, Proposition 2 below provides the value function and the optimal speed of trading in closed-form. Moreover, in Subsection 4.1 we use an expansion method to approximate the value function when the investor is ambiguity averse to both the diffusion and jump factors of the model for the midprices of electricity contracts.

Proposition 2. ***Trading under diffusion ambiguity.***
In the limit $\varepsilon \downarrow 0$ and $\phi > 0$, so that the investor is ambiguity averse only to the diffusive component of the reference measure $\mathbb{P}$, the HJBI in (20) reduces to

$$\begin{aligned}(\partial_t + \mathcal{L})\,H + \frac{1}{4\,a}\left[(b_1\partial_{P^1} - b_2\partial_{P^2} + \partial_q)\,H + \left(P^2 - P^1\right)\right]^2 - \frac{1}{2}\,\boldsymbol{\mathcal{D}}H'\,\boldsymbol{\Omega}'\,\boldsymbol{\Phi}^{-1}\,\boldsymbol{\Omega}\,\boldsymbol{\mathcal{D}}H \\ + \sum_{i=1}^{2}\lambda_i\int_{-\infty}^{\infty}\Delta H_i\,G_i(dy) \;=\; 0\,. \end{aligned} \tag{23}$$

Moreover, this HJBI equation admits a solution of the form

$$H(t,\boldsymbol{P},q) = \ell_0(t) + \boldsymbol{\ell}_0^\intercal(t)\,\boldsymbol{P} + \boldsymbol{P}^\intercal\boldsymbol{\ell}_{01}\boldsymbol{P} + \left(\ell_{10}(t) + (P^1 - P^2) + \boldsymbol{\ell}_1^\intercal(t)\boldsymbol{P}\right)\,q + \ell_2(t)\,q^2\,, \tag{24}$$

where $\boldsymbol{\ell}_0(t) = \begin{pmatrix}\ell_{01}(t)\\ \ell_{02}(t)\end{pmatrix}$, $\boldsymbol{\ell}_{01}(t) = \begin{pmatrix}\ell_{021}(t) & \frac{1}{2}\ell_{012}(t)\\ \frac{1}{2}\ell_{012}(t) & \ell_{022}(t)\end{pmatrix}$, $\boldsymbol{\ell}_1(t) = \begin{pmatrix}\ell_{11}(t)\\ \ell_{12}(t)\end{pmatrix}$ *are vector and matrix-valued deterministic functions of time. In this case, the optimal speed of trading is*

$$\nu^* = \frac{1}{2\,a}\left(\mathfrak{l}_0^*(t) + \mathfrak{l}_1^*(t)\,P^1 + \mathfrak{l}_2^*(t)\,P^2 + \mathfrak{l}_3^*(t)\,q\right)\,, \tag{25}$$

and the optimal drift adjustments are

$$
\begin{aligned}
\eta^*_{1t} &= -\phi\,\sigma_1\Big(\ell_{01}(t)+\rho\,\ell_{02}(t)+\left(2\,\ell_{021}(t)+\rho\,\ell_{012}(t)\right)\,P^1\\
&\qquad\qquad +\left(\ell_{012}(t)+2\,\rho\,\ell_{022}(t)\right)\,P^2+\left(\ell_{11}(t)+\rho\,\ell_{12}(t)+1-\rho\right)q\Big)\,,\\
\eta^*_{2t} &= -\phi\,\sigma_2\Big(\rho\,\ell_{01}(t)+\ell_{02}(t)+\left(2\,\rho\,\ell_{021}(t)+\ell_{012}(t)\right)\,P^1\\
&\qquad\qquad +\left(\rho\,\ell_{012}(t)+2\,\ell_{022}(t)\right)\,P^2+\left(\rho\,\ell_{11}(t)+\ell_{12}(t)+\rho-1\right)q\Big)\,,
\end{aligned}
\tag{26}
$$

where $\mathfrak{l}^*_0, \mathfrak{l}^*_1, \mathfrak{l}^*_2$ *and* $\mathfrak{l}^*_3$ *have the affine structure:*

$$\mathfrak{l}^*_0(t) = b_1\ell_{01}(t) - b_2\ell_{02}(t) + \ell_{10}(t)\,, \tag{27a}$$

$$\mathfrak{l}^*_1(t) = 2b_1\ell_{021}(t) - b_2\ell_{012}(t) + \ell_{11}(t)\,, \tag{27b}$$

$$\mathfrak{l}^*_2(t) = b_1\ell_{012}(t) - 2b_2\ell_{022}(t) + \ell_{12}(t)\,, \tag{27c}$$

$$\mathfrak{l}^*_3(t) = b_1(\ell_{11}(t)+1) - b_2(\ell_{12}(t)-1) + 2\ell_2(t)\,, \tag{27d}$$

and $\ell_0(t), \ell_{01}(t), \cdots, \ell_2(t)$ *are deterministic functions of time, which solve the ODE system* (7.2) *shown in the Appendix.*

Proof. See Appendix (7.2) for the proof.

We use the results in Proposition 2 to obtain the dynamics of midprices when the investor is ambiguous to only the diffusive factor, as shown in the following Corollary.

Corollary 1. ***Midprice dynamics under diffusion ambiguity.***
In the limit $\varepsilon \downarrow 0$ *and* $\phi > 0$*, so that the investor is ambiguous to only the diffusive component, the midprices satisfy the following stochastic differential equations (SDEs) under the optimal measure* $\mathbb{Q}^{\boldsymbol{\eta}^*}$*:*

$$dP^{1,\nu}_t = \kappa_1\left(\theta^1_t - \tfrac{\phi}{\kappa_1}\,\sigma_1\left(\partial_{P^1}+\rho\,\partial_{P^2}\right)H(t,\boldsymbol{P}_t,q_t) - P^{1,\nu}_t\right)dt + b_1\,\nu_t\,dt + \sigma_1\,dW^{1*}_t + \,dJ^1_t\,, \tag{28a}$$

$$dP^{2,\nu}_t = \kappa_2\left(\theta^2_t - \tfrac{\phi}{\kappa_2}\,\sigma_2\left(\rho\,\partial_{P^1}+\partial_{P^2}\right)H(t,\boldsymbol{P}_t,q_t) - P^{2,\nu}_t\right)dt - b_2\,\nu_t\,dt + \sigma_2\,dW^{2*}_t + \,dJ^2_t\,, \tag{28b}$$

where $W^{i*} = (W^{i*}_t)_{\{0\le t\le T\}}$ *are standard* $\mathbb{Q}^{\eta^*}$*-Brownian motions with correlation* ρ*, and*

$$\partial_{P^1} H(t, \boldsymbol{P}, q) = l^*_{01} + 2\, l^*_{021}\, P^1 + l^*_{012}\, P^2 + (l^*_{11} + 1)\, q\,, \tag{29a}$$

$$\partial_{P^2} H(t, \boldsymbol{P}, q) = l^*_{02} + 2\, l^*_{022}\, P^2 + l^*_{012}\, P^1 + (l^*_{12} - 1)\, q\,. \tag{29b}$$

Proof. Substitute (26) in the Radon-Nikodym derivative (9) and a straight forward application of Girsanov's theorem allows us to write the $\mathbb{P}$-Brownian motions in terms of $\mathbb{Q}^{\eta^*}$-Brownian motions. Then, rewriting the SDEs in terms of these new Brownian motions leads to (28). ∎

Thus, when $\varepsilon \downarrow 0$ and $\phi > 0$ the investor rejects the reference model in favor of one where midprices mean-revert to a level that consists of the seasonal trend in the reference model plus the new term $-\frac{\phi}{\kappa_1}\, \sigma_1\, (\partial_{P^1} H + \rho\, \partial_{P^2} H)$ for midprices in location 1 and a similar expression for midprices in location 2. The new term that appears in the drift of midprices is proportional to the ambiguity aversion parameter ϕ and to the volatility of the diffusive component σ_i.

4.1. Asymptotic Analysis First Order

The HJBI (20) is nonlinear and we cannot obtain a solution in closed-form, so we employ perturbation methods to approximate the value function employing the expansion

$$H(t, \boldsymbol{P}, q) = H_0(t, \boldsymbol{P}, q) + \phi\, H_D(t, \boldsymbol{P}, q) + \varepsilon\, H_J(t, \boldsymbol{P}, q) + O(\upsilon)\,, \tag{30}$$

where $\upsilon = \max(\phi^2,\, \phi\,\varepsilon,\, \varepsilon^2)$. The following three Propositions provide closed-form solutions for each term in the right-hand side of (30).

Proposition 3. *In the limit* $(\varepsilon, \phi) \downarrow (0, 0)$*, the value function of the ambiguity neutral investor, which is denoted by* $H_0(t, \boldsymbol{P}, q)$*, satisfies the partial integro-differential equation (PIDE)*

$$\begin{aligned} 0 = (\partial_t + \mathcal{L})\, H_0 + \tfrac{1}{4\,a}\, \left[(b_1\, \partial_{P^1} - b_2\, \partial_{P^2} + \partial_q)\, H_0 + \left(P^2 - P^1\right)\right]^2 \\ + \sum_{i=1,2} \lambda_i \int_{-\infty}^{\infty} \Delta_i(y) H_0\, G_i(dy)\,, \end{aligned} \tag{31}$$

subject to the terminal condition $H_0(T, P^1, P^2, q) = (P^1 - P^2)\, q - \alpha\, q^2$*, and admits the ansatz*

$$
\begin{aligned}
H_0(t, \boldsymbol{P}, q) = \; & \ell_0^{(0)}(t) + \boldsymbol{\ell}_0^{(0)\intercal}(t)\, \boldsymbol{P} + \boldsymbol{P}^\intercal \boldsymbol{\ell}_{01}^{(0)} \boldsymbol{P} \\
& + \left(\ell_{10}^{(0)}(t) + (P^1 - P^2) + \boldsymbol{\ell}_1^{(0)\intercal}(t) \boldsymbol{P}\right)\, q + \ell_2(t)\, q^2 \,,
\end{aligned}
\tag{32}
$$

where $\boldsymbol{\ell}_0^{(0)}(t) = \begin{pmatrix} \ell_{01}^{(0)}(t) \\ \ell_{02}^{(0)}(t) \end{pmatrix}$, $\boldsymbol{\ell}_{01}^{(0)}(t) = \begin{pmatrix} \ell_{021}^{(0)}(t) & \frac{1}{2}\ell_{012}^{(0)}(t) \\ \frac{1}{2}\ell_{012}^{(0)}(t) & \ell_{022}^{(0)}(t) \end{pmatrix}$, $\boldsymbol{\ell}_1^{(0)}(t) = \begin{pmatrix} \ell_{11}^{(0)}(t) \\ \ell_{12}^{(0)}(t) \end{pmatrix}$ *are vector and matrix-valued deterministic functions of time, which satisfy the ODE system provided in Appendix 7.3.*

Proof. see Appendix (7.3) the proof. ∎

Proposition 4. *Let* $H(t, \boldsymbol{P}, q)$ *be as in the expansion* (30)*, then* $H_D(t, \boldsymbol{P}, q)$ *satisfies*

$$
\begin{aligned}
& (\partial_t + \mathcal{L})\, H_D + f_D(t, \boldsymbol{P}, q)\, (b_1\, \partial_{P^1} - b_2\, \partial_{P^2} + \partial_q)\, H_D + \sum_{i=1}^{2} \lambda_i \int_{-\infty}^{\infty} \Delta_i(y) H_D\, G_i(dy) \\
& \qquad - \frac{1}{2\phi}\, \boldsymbol{\mathcal{D}} H_0'\, \boldsymbol{\Omega}'\, \boldsymbol{\Phi}^{-1}\, \boldsymbol{\Omega}\, \boldsymbol{\mathcal{D}} H_0 = 0 \,,
\end{aligned}
\tag{33}
$$

with terminal condition $H_D(T, P^1, P^2, q) = 0$*, where*

$$
f_D(t, \boldsymbol{P}, q) = \frac{1}{2\,a} \left[(b_1\, \partial_{P^1} - b_2\, \partial_{P^2} + \partial_q)\, H_0(t, \boldsymbol{P}, q) + \left(P^2 - P^1\right) \right] .
$$

Equation (33) *admits the ansatz*

$$
\begin{aligned}
H_D(t, \boldsymbol{P}, q) = \; & \ell_0^{(0)}(t) + \boldsymbol{\ell}_0^{(0)\intercal}(t)\, \boldsymbol{P} + \boldsymbol{P}^\intercal \boldsymbol{\ell}_{01}^{(0)} \boldsymbol{P} \\
& + \left(\ell_{10}^{(0)}(t) + \boldsymbol{\ell}_1^{(0)\intercal}(t) \boldsymbol{P}\right)\, q + \ell_2(t)\, q^2 \,,
\end{aligned}
\tag{34}
$$

where $\boldsymbol{\ell}_0^{(0)}(t) = \begin{pmatrix} \ell_{01}^{(0)}(t) \\ \ell_{02}^{(0)}(t) \end{pmatrix}$, $\boldsymbol{\ell}_{01}^{(0)}(t) = \begin{pmatrix} \ell_{021}^{(0)}(t) & \frac{1}{2}\ell_{012}^{(0)}(t) \\ \frac{1}{2}\ell_{012}^{(0)}(t) & \ell_{022}^{(0)}(t) \end{pmatrix}$, $\boldsymbol{\ell}_1^{(0)}(t) = \begin{pmatrix} \ell_{11}^{(0)}(t) \\ \ell_{12}^{(0)}(t) \end{pmatrix}$ *are vector and matrix-valued deterministic functions of time, which satisfy the ODE system provided in Appendix 7.4.*

Proof. See Appendix (7.4) for the proof. ∎

Proposition 5. *Let $H(t,\boldsymbol{P},q)$ be as in the expansion* (30)*, then $H_J(t,\boldsymbol{P},q)$ satisfies*

$$\begin{aligned}(\partial_t+\mathcal{L})H_J &+ f_H(t,\boldsymbol{P},q)\ (b_1\,\partial_{P^1}-b_2\,\partial_{P^2}+\partial_q)\ H_J \\ &+\sum_{i=1}^{2}\lambda_i\int_{-\infty}^{\infty}\left(\Delta_i(y)H_J-\frac{1}{2}(\Delta_i(y)H_0)^2\right)\,G_i(dy)=0\,,\end{aligned}\tag{35}$$

subject to the terminal condition $H_J(T,\boldsymbol{P},q)=0$, where

$$f_H(t,\boldsymbol{P},q)=\frac{1}{2\,a}\left[(b_1\,\partial_{P^1}-b_2\,\partial_{P^2}+\partial_q)\,H_0(t,\boldsymbol{P},q)+\left(P^2-P^1\right)\right]\,.$$

Equation (35) *admits the ansatz*

$$\begin{aligned}H_J(t,\boldsymbol{P},q)=\ &\ell_0^{(0)}(t)+\boldsymbol{\ell}_0^{(0)\intercal}(t)\,\boldsymbol{P}+\boldsymbol{P}^\intercal\boldsymbol{\ell}_{01}^{(0)}\boldsymbol{P}\\ &+\left(\ell_{10}^{(0)}(t)+\boldsymbol{\ell}_1^{(0)\intercal}(t)\boldsymbol{P}\right)\,q+\ell_2(t)\,q^2\,,\end{aligned}\tag{36}$$

where $\boldsymbol{\ell}_0^{(0)}(t)=\begin{pmatrix}\ell_{01}^{(0)}(t)\\ \ell_{02}^{(0)}(t)\end{pmatrix}$, $\boldsymbol{\ell}_{01}^{(0)}(t)=\begin{pmatrix}\ell_{021}^{(0)}(t) & \frac{1}{2}\ell_{012}^{(0)}(t)\\ \frac{1}{2}\ell_{012}^{(0)}(t) & \ell_{022}^{(0)}(t)\end{pmatrix}$, $\boldsymbol{\ell}_1^{(0)}(t)=\begin{pmatrix}\ell_{11}^{(0)}(t)\\ \ell_{12}^{(0)}(t)\end{pmatrix}$ *are vector and matrix-valued deterministic functions of time, which satisfy the ODE system provided in Appendix 7.5.*

Proof. See Appendix (7.5) for the proof. ■

Proposition 6. *Given the first order expansion of* (30)*, the residual function $R^{\phi,\varepsilon}(t,\boldsymbol{P},q)$ satisfies the following PIDE:*

$$\begin{aligned}&(\partial_t+\mathcal{L})R^{\phi,\epsilon}+\frac{1}{4\,a}\left[(\mathcal{L}_1R^{\phi,\varepsilon})^2+2\,\mathcal{L}_0H_0\right]+M_1(H_0,H_D,H_J,\varepsilon,\phi)\,R^{\phi,\varepsilon}\\ &+M_2(H_0,H_D,H_J,\varepsilon,\phi)\,\boldsymbol{\Omega}'\,\boldsymbol{\Phi}^{-1}\,\boldsymbol{\Omega}\,\boldsymbol{\mathcal{D}}R^{\phi,\varepsilon}-\frac{1}{2}\,(\boldsymbol{\mathcal{D}}R^{\phi,\varepsilon})'\,\boldsymbol{\Omega}\,\boldsymbol{\Phi}^{-1}\,\boldsymbol{\Omega}\boldsymbol{\mathcal{D}}R^{\phi,\varepsilon}\\ &+\sum_{i=1}^{2}\lambda_i\int_{-\infty}^{\infty}\left(\Delta_i(y)R^{\phi,\varepsilon}-\frac{\varepsilon}{2}\,(\Delta_i(y)\,R^{\phi,\epsilon})^2+M_3(\Delta_i(y)H_0,\Delta_i(y)H_D,\Delta_i(y)H_J,\phi,\varepsilon)\,\Delta_i(y)R^{\phi,\varepsilon}\right)\,G_i(dy)\\ &+M_4(H_0,H_D,H_J,\varepsilon,\phi)+M_5(\Delta_i(y)H_0,\Delta_i(y)H_D,\Delta_i(y)H_J,\phi,\varepsilon)+o(H)=0\,,\end{aligned}\tag{37}$$

with the terminal condition

$$R^{\phi,\varepsilon}(T,\boldsymbol{P},q)=0\,,\qquad\forall P^1,\ P^2,\ q\,,$$

where M_1, M_2, M_3, M_4, M_5 are given by:

$$
\begin{aligned}
M_1 &= \frac{1}{2\,a}\mathcal{L}_1(\phi\, H_D + \varepsilon\, H_J)\,,\\
M_2 &= -\boldsymbol{\mathcal{D}}(H_0 + \phi\, H_D + \varepsilon\, H_J)'\,,\\
M_3 &= -\varepsilon\,\Delta_i(y)H_0 - \phi\,\varepsilon\,\Delta_i(y)H_D - \varepsilon^2\,\Delta_i(y)H_J\,,\\
M_4 &= \phi^2(\mathcal{L}_1 H_D)^2 + \varepsilon^2\,(\mathcal{L}_1 H_J)^2 + 2\,\phi\,\varepsilon\,(\mathcal{L}_1 H_D)\,(\mathcal{L}_1 H_J)\\
&\quad - \frac{1}{2}\,\boldsymbol{\mathcal{D}}(\phi\, H_D + \varepsilon\, H_J)'\boldsymbol{\Omega}\,\boldsymbol{\Phi}^{-1}\,\boldsymbol{\Omega}\,\boldsymbol{\mathcal{D}}(\phi\, H_D + \varepsilon\, H_J) - (\boldsymbol{\mathcal{D}}H_0)'\boldsymbol{\Omega}\,\boldsymbol{\Phi}^{-1}\,\boldsymbol{\Omega}\boldsymbol{\mathcal{D}}(\phi\, H_D + \varepsilon\, H_J)\,,\\
M_5 &= -\frac{\varepsilon}{2}(\phi^2\,(\Delta_i(y)H_D)^2 + \varepsilon^2\,(\Delta_i(y)H_J)^2 + 2\,\phi\,(\Delta_i(y)H_0)\,(\Delta_i(y)H_D)\\
&\quad + 2\,\varepsilon\,(\Delta_i(y)H_0)\,(\Delta_i(y)H_J) + 2\,\varepsilon\,\phi\,(\Delta_i(y)H_D)\,(\Delta_i(y)H_J))\,.
\end{aligned}
\tag{38}
$$

Proof. Substitute (30) into (20) and using the PIDEs (31), (33) and (35), we obtain the above PIDE satisfied by $R^{\phi,\varepsilon}(t, \boldsymbol{P}, q)$. ■

5. Trading Across locations: ambiguity aversion effects and performance of strategy

In this section we show how ambiguity aversion affects the investor's model for the midprice dynamics and illustrate the financial performance of the trading strategy. Specifically, in Subsection 5.1 we assume that the investor is confident about the jump factor in the reference model (i.e., $\varepsilon \downarrow 0$) and show how ambiguity specific to the diffusive factor affects the investor's model for the midprices of electricity contracts. In Subsection 5.2 we employ simulations to illustrate the investor's trading strategy under model ambiguity to the diffusive and jump factors, and show the strategy's financial performance under different levels of ambiguity aversion.

In the simulations, the midprices of contracts are simulated under the statistical measure $\widetilde{\mathbb{P}}$. Under this measure, midprices satisfy the SDEs in (2), but the parameters are different from those employed by the investor in her reference measure $\mathbb{P}$.

Model	κ_1	κ_2	σ_1	σ_2	ρ	θ_1	θ_2	m_1^+	m_2^+	m_1^-	m_2^-	λ_1	λ_2	p_1	p_2
$\mathbb{P}$	3.5	3.5	1	1.5	0.5	20	20	0.7	0.7	0.3	0.3	2	2	0.6	0.6
$\widetilde{\mathbb{P}}$	3.0	2.5	1	1.25	0.6	20	20	0.6	0.6	0.4	0.4	2.5	2.5	0.65	0.65

Moreover, we assume that the distribution of jumps in both locations is the double exponential distribution:

$$G_i(dy) = \left\{ p_i \, m_i^+ \, e^{-m_i^+ y} \, \mathbb{1}_{z>0} + (1-p_i) \, m_i^- \, e^{-m_i^- |y|} \, \mathbb{1}_{y\leq 0} \right\} \, dy \,, \tag{39}$$

where $m_i^{\pm} > 0$, λ_i are the arrival rate of jumps, and p_i are the probabilities of upwards jumps in midprices in each location.

Table 5 shows the parameters estimated by the investor and those used to simulate prices. Finally, the permanent price impact parameters are $b_1 = 1\times 10^{-6}$, $b_2 = 2\times 10^{-6}$; the aggregate temporary price impact parameter are $a = a_1 + a_2 = 3 \times 10^{-6}$; and the terminal liquidation inventory penalty parameter is $\alpha = 1000 \times a$.

5.1. Effect of ambiguity aversion to diffusive factor

When the investor is ambiguous to only the diffusive factor, we obtain the optimal measure in closed-form, and the midprices of the electricity contracts satisfy the SDEs (28a) and (28b), see Corollary 1, which we repeat here for convenience:

$$\begin{aligned} dP_t^{1,\nu} &= \kappa_1 \left(\theta_t^1 - \tfrac{\phi}{\kappa_1} \, \sigma_1 \, (\partial_{P_1} + \rho \, \partial_{P_2}) \, H(t, \boldsymbol{P}_t, q_t) - P_t^{1,\nu} \right) \, dt + b_1 \, \nu_t \, dt + \sigma_1 \, dW_t^{1*} + dJ_t^1 \,, \\ dP_t^{2,\nu} &= \kappa_2 \left(\theta_t^2 - \tfrac{\phi}{\kappa_2} \, \sigma_2 \, (\rho \, \partial_{P_1} + \partial_{P_2}) \, H(t, \boldsymbol{P}_t, q_t) - P_t^{2,\nu} \right) \, dt - b_2 \, \nu_t \, dt + \sigma_2 \, dW_t^{2*} + dJ_t^2 \,. \end{aligned}$$

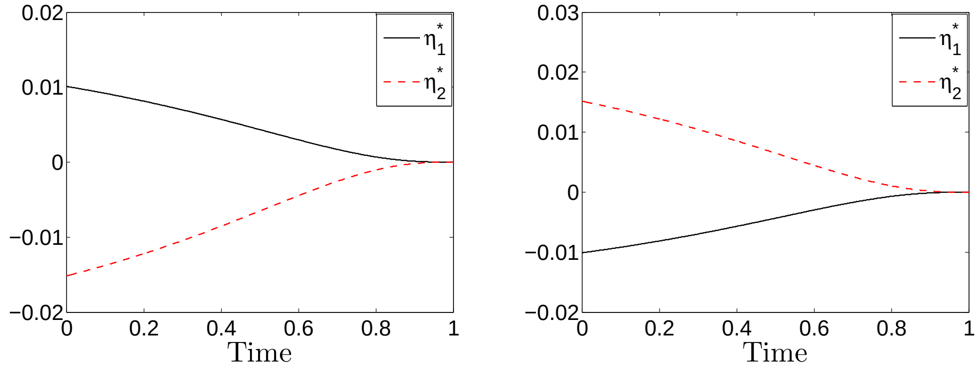

Figure 2: Optimal drift $\eta_t^{1*} = -\phi\,\sigma_1\,(\partial_{P_1} H + \rho\,\partial_{P_2} H)$, $\eta_t^{2*} = -\phi\,\sigma_2\,(\rho\,\partial_{P_1} + \partial_{P_2})\,H$, Left Panel: $P^1 = 19$, $P^2 = 21$, Right Panel: $P^1 = 21$, $P^2 = 19$, $Q = 0$, $\phi = 10^{-6}$.

To gain insights into how the investor modifies the dynamics of the reference model to incorporate ambiguity aversion, Figure 2 shows the optimal drift adjustments $\eta_t^{1*} = -\phi\,\sigma_1\,(\partial_{P_1} + \rho\,\partial_{P_2})\,H(t, \boldsymbol{P}_t, q_t)$ and $\eta_t^{2*} = -\phi\,\sigma_2\,(\rho\,\partial_{P_1} + \partial_{P_2})\,H(t, \boldsymbol{P}_t, q_t)$, for fixed prices $P^1 = 19$ and $P^2 = 21$. The figure shows that midprices revert to the seasonal trend θ_t^i under the optimal measure faster than they would under the investor's reference model. For example, if the price in location 1 is below the seasonal trend $\theta^1 = 20$, then the optimal drift is positive, so it adds an upward pressure in the drift of the price $P_t^{1,\nu}$. Similarly, if the price is above its seasonal trend, the drift in the optimal measure is negative, so the price of electricity is pulled down quicker to the seasonal level of the reference model.

Moreover, from the dynamics of $P_t^{1,\nu}$ and $P_t^{2,\nu}$ shown in equations (28a) and (28b) respectively, we see that the drift adjustment is proportional to the volatility of midprices and the ambiguity aversion parameter. Recall that $\sigma_1 = 1$ and $\sigma_2 = 1.5$, which explains why the drift adjustment for midprices in location 2 is larger than that in location 1. Moreover, it is also straightforward to see that the more ambiguous (i.e. higher ϕ) is the investor to the diffusive component, the stronger is the effect on the drift to pull midprices to the seasonal trend θ_t^i.

5.2. Strategy's performance: effect of ambiguity aversion to diffusive and jump factor

Here we employ simulations to show how ambiguity aversion to the diffusive and jump factor affects the investor's trading behaviour and the financial performance of the trading strategy. We use the expansion solution for the value function derived in Subsection 4.1 to compute the investor's speed of trading. Recall that the simulations of price paths are performed under the statistical measure $\widetilde{\mathbb{P}}$.

To gain insights into how ambiguity aversion affects the performance of the strategy we consider three cases. Case 1: the investor is ambiguous to only the diffusive factor. Case 2: the investor is ambiguous to only the jump factor. Case 3: the investor is ambiguous to the diffusive and jump factor.

For each case we show: (i) sample midprice paths under the measure $\widetilde{\mathbb{P}}$, (ii) trading speed for different levels of ambiguity aversion, (iii) percentiles of the inventory held by the investor throughout the life of the strategy (performing 10,000 simulations in each instance), and (iv) the Sharpe ratio of the strategy for different levels of ambiguity aversion. The Sharpe ratio is calculated as the average profit of the strategy divided by the standard deviation of the profit (risk-free rate is zero).

In the three cases described below the effect of model uncertainty is to make the trading strategy more conservative than that resulting from the model without ambiguity aversion. Specifically, everything else equal, ambiguity aversion slows down the trading speed of contracts in both locations. Thus, compared to the case $(\varepsilon, \phi) \downarrow (0, 0)$ (i.e. no model uncertainty) the less confident the investor is about the reference model, the less inventory she holds throughout the life of the strategy.

Furthermore, for a range of parameter values of ambiguity aversion, as the strategy becomes more conservative, the Sharpe ratio increases. Clearly, there is a tradeoff between the expected profits of the strategy and the volatility of the profits. As the investor becomes more ambiguous about the reference model, she reduces the quantity of contracts she is willing to hold in both locations and this has an effect on the expected and volatility of profits.

Case 1: $\varepsilon \downarrow 0$ and $\phi \geq 0$.

Here the investor is ambiguous to only the diffusive factor. Figure 3 shows: one simulation of prices in both locations, the investor's speed of trading (in location 1) for three levels of ambiguity aversion specific to the diffusive factors, and the inventory path in location 1.

The second panel in Figure 3 shows that as the investor becomes more ambiguity averse, the speed at which she trades, ceteris paribus, slows down. Intuitively, as the investor is less certain about her reference model, she becomes more conservative by trading less contracts.

The third picture in the figure shows the path of inventory holdings. Clearly, as the investor becomes more ambiguity averse, and slows down the speed of trading, the amount of contracts held at any point in time is lower the less confident the investor is about her model.

Finally, Figure 4 shows the Sharpe ratio of the strategy and percentiles of the inventory holdings using 10,000 simulations. As the investor becomes more conservative, as a result of model uncertainty, she trades less and holds less inventory, all of which has an effect on the mean and the standard deviation of the profit and loss (PnL) of the strategy. We measure the effect on the PnL by computing the strategy's Sharpe ratio (mean of PnL divided by volatility of PnL); this is depicted in the left-hand side picture of Figure 4 for a range of values of the ambiguity aversion ϕ. Moreover, the right-hand side of the figure shows the percentiles of the inventory holdings for different levels of ambiguity aversion to the diffusive factor.

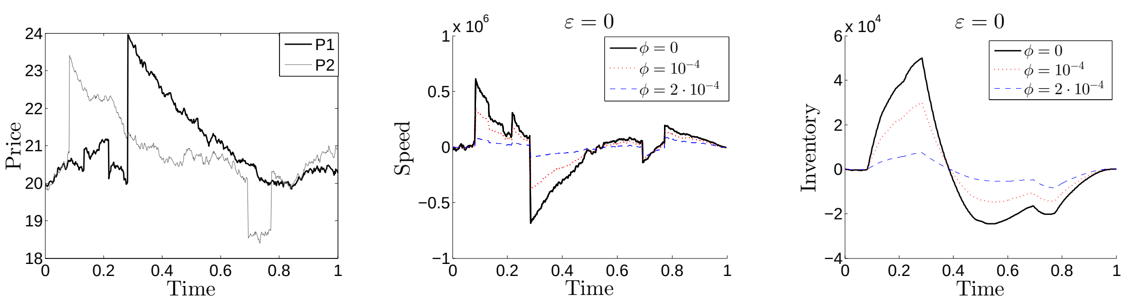

Figure 3: Midprice paths, speed of trading, inventory holdings in location 1. Different values of ϕ and $\varepsilon \downarrow 0$

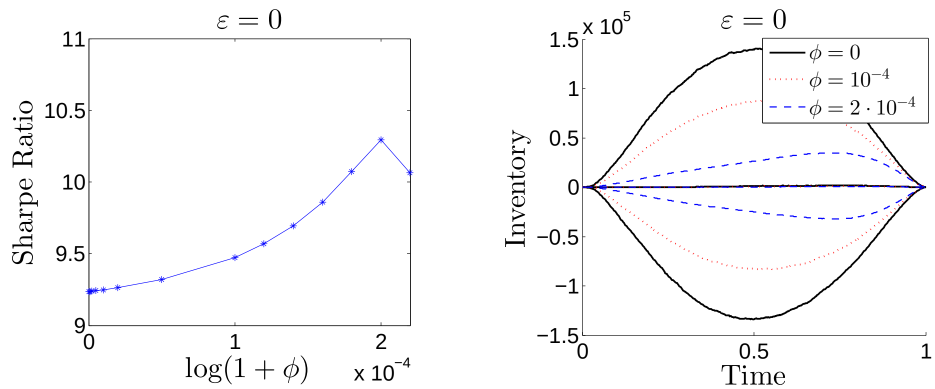

Figure 4: Sharpe ratios and percentiles of inventory holdings in location 1 for different values of ϕ and $\varepsilon \downarrow 0$

Case 2: $\varepsilon \geq 0$ *and* $\phi \downarrow 0$.

Here the investor is ambiguous to only the jump factor. Figure 5 shows: one simulation of prices in both locations, the investor's speed of trading (in location 1) for three levels of ambiguity aversion specific to the jump factors, and the inventory path in location 1. The interpretation of the results is similar to that in Case 1.

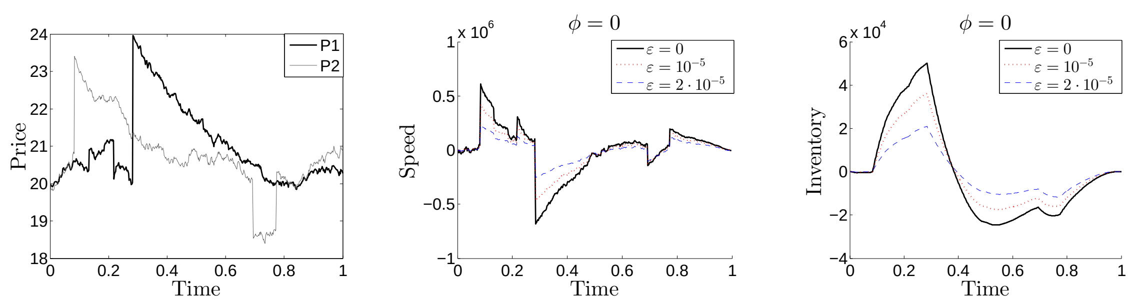

Figure 5: Midprice paths, speed of trading, inventory holdings in location 1. Different values of ε and $\phi \downarrow 0$

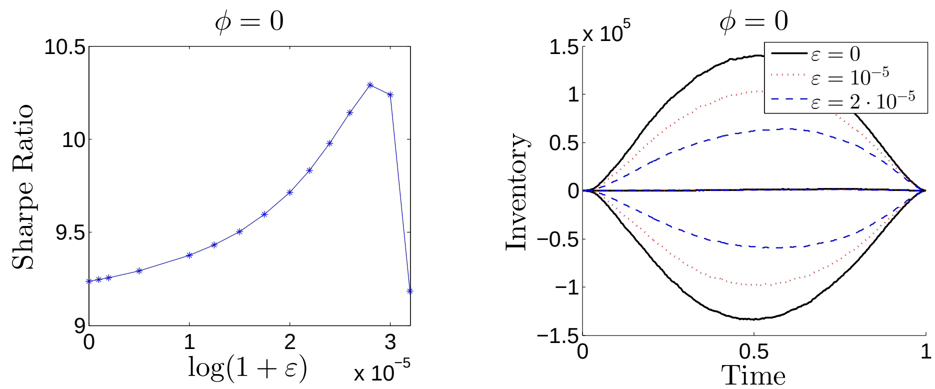

Figure 6: Sharpe ratios and percentiles of inventory holdings in location 1 for different values of ε and $\phi \downarrow 0$

Case 3: $\varepsilon \geq 0$ *and* $\phi \geq 0$.

Here we show how the strategy performs when the investor is ambiguous to both factors. The left-hand side of Figure 7 shows the Sharpe ratio for fixed $\phi = 10^{-5}$ and different values of the jump ambiguity parameter ε. Similarly, the right-hand side of Figure 7 shows the Sharpe ratio for fixed $\varepsilon = 10^{-5}$ and different values of the diffusive ambiguity parameter ϕ.

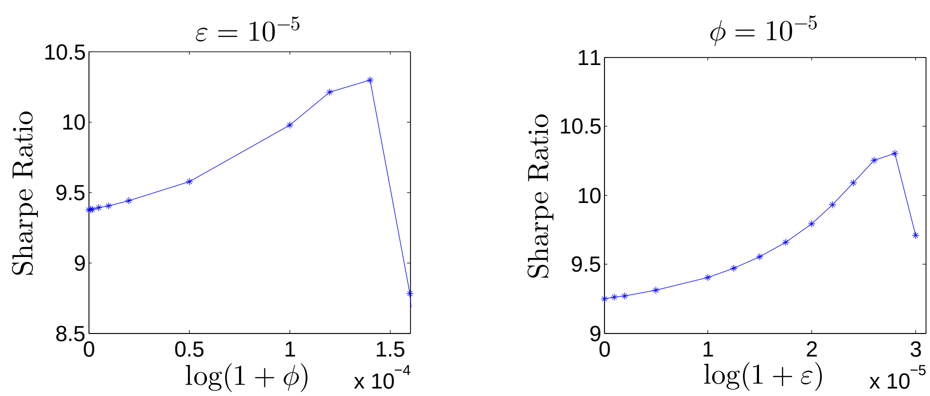

Figure 7: Sharpe Ratio for different values $\varepsilon > 0$, $\phi > 0$

6. Conclusions

We show how an ambiguity averse investor trades electricity contracts in two locations joined by an interconnector. The investor employs a model that incorporates the price impact of her trades: her trading activity has a permanent effect on the prices of the contracts in both locations, and the prices she receives are worse than the prevailing quoted prices.

The investor acknowledges that her model of prices may be misspecified. She deals with this model uncertainty by considering alternative models when developing the optimal trading strategy. We show that the effect of model uncertainty is to make the trading strategy more conservative than that resulting from the model without ambiguity aversion. Specifically, everything else equal, ambiguity aversion slows down the trading speed of contracts in both locations and the investor holds less inventory throughout the life of the strategy. Finally, we show that for a range of parameter values of ambiguity aversion, as the strategy becomes more conservative, the Sharpe ratio of the strategy (mean PnL to volatility of PnL) increases.

7. Appendix

7.1. Proof of the Verification theorem

Proof. Let $\widehat{H}(t,\boldsymbol{P},q)$ be the solution of (17). We must show that $\widehat{H}$ is also the value function as described in (6). Employing Ito's lemma we obtain

$$\begin{aligned}\widehat{H}(t,P^1_{T^-},P^2_{T^-},q) = \widehat{H}(t,\boldsymbol{P},q) &+ \int_t^T (\partial_t+\mathcal{L}^\nu)\widehat{H}(u,P^1_u,P^2_u,q_u)\,du\\ &+ \int_t^T \mathcal{D}H'\boldsymbol{\Omega}\,dW_u + \sum_{i=1}^{2}\int_{-\infty}^{\infty}\int_t^T \Delta_i(y)H\,\mu_i(dy,dt)\,,\end{aligned}$$

for any admissible trading control ν_t, take expectations over any admissible measure control pairs $(\boldsymbol{\eta},\boldsymbol{g})$, we obtain

$$\begin{aligned}\mathbb{E}^{(\boldsymbol{\eta},\boldsymbol{g})}_{t,\boldsymbol{P},q}\left[\widehat{H}(t,P^1_{T^-},P^2_{T^-},q)\right] = \widehat{H}(t,\boldsymbol{P},q) &+ \mathbb{E}^{(\boldsymbol{\eta},\boldsymbol{g})}_{t,\boldsymbol{P},q}\left[\int_t^T (\partial_t+\mathcal{L}^\nu)H(u,P^1_u,P^2_u,q_u)\,du\right]\\ &+ \mathbb{E}^{(\boldsymbol{\eta},\boldsymbol{g})}_{t,\boldsymbol{P},q}\left[\int_t^T \mathcal{D}H'\boldsymbol{\Omega}(\,dW_u+\boldsymbol{\eta}\,dt)\right]\\ &+ \sum_{i=1}^{2}\mathbb{E}^{(\boldsymbol{\eta},\boldsymbol{g})}_{t,\boldsymbol{P},q}\left[\int_{-\infty}^{\infty}\int_t^T \Delta_i(y)H\,\mu_i(dy,dt)\right]\,,\end{aligned}$$

and since $\widehat{H}(t,\boldsymbol{P},q)$ is the solution of (17), we obtain

$$\begin{aligned}0 \leq\, & \partial_t \widehat{H} + \left(\mathcal{L}^{\nu} + f\left(t,\boldsymbol{P},\nu\right)\right)\widehat{H} \\ & + \sum_{i=1}^{2} \lambda_i \int_{-\infty}^{\infty} \left\{ \Delta_i(y) H\, e^{g_i(y)} + \frac{1}{\varepsilon}\left(1 + e^{g_i(y)}\left(g_i(y)-1\right)\right)\right\} G_i(dy) \\ & + \left\{ \boldsymbol{\eta}'\, \boldsymbol{\Omega}\, \boldsymbol{\mathcal{D}} \boldsymbol{H} + \frac{1}{2} \boldsymbol{\eta}'\, \boldsymbol{\Phi}^{-1}\, \boldsymbol{\eta} \right\},\end{aligned}$$

Moreover, use the fact

$$\mathbb{E}^{(\boldsymbol{\eta},\boldsymbol{g})}_{t,\boldsymbol{P},q}\left[\int_{-\infty}^{\infty}\int_t^T \Delta_i(y) H\, \mu_i(dy,dt)\right] = \lambda_i \int_{-\infty}^{\infty}\int_t^T \Delta_i(y) H\, e^{g_i(y)} G_i(dy) dt$$

to write

$$\begin{aligned}\widehat{H}(t,\boldsymbol{P},q) \leq\, & \mathbb{E}^{(\boldsymbol{\eta},\boldsymbol{g})}_{t,\boldsymbol{P},q}\left[\widehat{H}(t,P^1_{T^-},P^2_{T^-},q)\right] + \mathbb{E}^{(\boldsymbol{\eta},\boldsymbol{g})}_{t,\boldsymbol{P},q}\left[\int_t^T f(t,\boldsymbol{P},\nu)\widehat{H}\right] \\ & + \sum_{i=1}^{2} \lambda_i \int_{-\infty}^{\infty} \left\{ \frac{1}{\varepsilon}\left(1 + e^{g_i(y)}\left(g_i(y)-1\right)\right)\right\} G_i(dy) + \frac{1}{2} \boldsymbol{\eta}'\, \boldsymbol{\Phi}^{-1}\, \boldsymbol{\eta}\,.\end{aligned}$$

Using the terminal condition, we see that

$$\widehat{H}(t,\boldsymbol{P},q) \leq \mathbb{E}^{(\boldsymbol{\eta},\boldsymbol{g})}_{t,\boldsymbol{P},q}\left[\int_t^T f(t,\boldsymbol{P},\nu)\, du + g(T,\boldsymbol{P},q) + \mathcal{H}(\mathbb{Q}|\mathbb{P})\right], \quad \forall (\boldsymbol{\eta},\boldsymbol{g})\,,$$

and obtain

$$\widehat{H} \leq \sup_{\nu} \inf_{Q\in\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}\left[\int_t^T f(s,P^1_s,P^2_s,\nu_s)\, ds\; + g(T,P^1_T,P^2_T,q_T) + \mathcal{H}(\mathbb{Q}|\mathbb{P}) \,|\, \mathcal{F}_t\right] = H\,.$$

To get the other part of the inequality, we fix $(\boldsymbol{\eta}^*,\boldsymbol{g}^*)$, for any admissible trading control ν, so that

$$\begin{aligned}0 \geq\, & \partial_t \widehat{H} + \left(\mathcal{L}^{\nu} + f\left(t,\boldsymbol{P},\nu\right)\right)\widehat{H} \\ & + \sum_{i=1}^{2} \lambda_i \int_{-\infty}^{\infty} \left\{ \Delta_i(y) H\, e^{g^*_i(y)} + \frac{1}{\varepsilon}\left(1 + e^{g^*_i(y)}\left(g^*_i(y)-1\right)\right)\right\} G_i(dy) \\ & + \left\{ \boldsymbol{\eta}^{*\prime}\, \boldsymbol{\Omega}\, \boldsymbol{\mathcal{D}} \boldsymbol{H} + \frac{1}{2} \boldsymbol{\eta}^{*\prime}\, \boldsymbol{\Phi}^{-1}\, \boldsymbol{\eta}^* \right\}.\end{aligned}$$

Thus

$$\widehat{H}(t,\boldsymbol{P},q) \geq \inf_{Q\in\mathbb{Q}} \mathbb{E}\left[\int_t^T f(t,\boldsymbol{P},\nu)\, du + g(T,\boldsymbol{P},q) + \mathcal{H}(\mathbb{Q}|\mathbb{P})\right], \quad \forall \nu\,,$$

such that

$$\widehat{H}(t,\boldsymbol{P},q) \geq \sup_{\nu} \inf_{Q\in\mathbb{Q}} \mathbb{E}\left[\int_t^T f(t,\boldsymbol{P},\nu)\, du + g(T,\boldsymbol{P},q) + \mathcal{H}(\mathbb{Q}|\mathbb{P})\right] = H\,.$$

This completes the proof. ■

7.2. Proof of Prop 2

Proof. To solve (23), we employ the ansatz (24). Collecting terms in powers of q, i.e. terms with factors $q^2, P^1\, q, \cdots, , P^2, P^1$, and constant terms, we obtain the following ODE system:

$$
\begin{aligned}
0 =&\partial_t \ell_2 + \frac{1}{a}\left[\frac{(b_1(\ell_{11}+1)+b_2(1-\ell_{12})}{2}+\ell_2\right]^2 \\
&- \frac{\varepsilon}{2}\left[\sigma_1^2(\ell_{11}+1)^2 + 2\rho\sigma_1\sigma_2(\ell_{11}+1)(\ell_{12}-1)+\sigma_2^2(\ell_{12}-1)^2\right]\,, \\
0 =&\partial_t \ell_{11} - \kappa_1(\ell_{11}+1) + \frac{1}{a}\left[\frac{(b_1(\ell_{11}+1)+b_2(1-\ell_{12})}{2}+\ell_2\right][2b_1\ell_{021}-b_2\ell_{02}+\ell_{11}] \\
&- \frac{\varepsilon}{2}\left[4\sigma_1^2\ell_{021}(\ell_{11}+1)+2\rho\sigma_1\sigma_2((\ell_{11}+1)\ell_{012}+2\ell_{021}(\ell_{12}-1)+2\sigma_2^2(\ell_{12}-1)\ell_{012}\right]\,, \\
0 =&\partial_t \ell_{12} - \kappa_2(\ell_{12}-1) + \frac{1}{a}\left[\frac{(b_1(\ell_{11}+1)+b_2(1-\ell_{12})}{2}+\ell_2\right][b_1\ell_{012}-2b_2\ell_{022}+\ell_{12}] \\
&- \frac{\varepsilon}{2}[2\sigma_1^2\ell_{012}(\ell_{11}+1)+2\rho\sigma_1\sigma_2(2(\ell_{11}+1)\ell_{022}+\ell_{012}(\ell_{12}-1)+4\sigma_2^2(\ell_{12}-1)\ell_{022}]\,, \\
0 =&\partial_t \ell_{10} + \kappa_1\theta_1(\ell_{11}+1)+\kappa_2\theta_2(\ell_{12}-1) + \frac{1}{a}\left[\frac{(b_1(\ell_{11}+1)+b_2(1-\ell_{12})}{2}+\ell_2\right][b_1\ell_{01}-b_2\ell_{02}+\ell_{10}] \\
&- \frac{\varepsilon}{2}[2\sigma_1^2\ell 01(\ell_{11}+1)+2\rho\sigma_1\sigma_2((\ell_{11}+1)\ell_{02}+\ell_{01}(\ell_{12}-1))+2\sigma_2^2(\ell_{12}-1)\ell_{02}]\,, \\
0 =&\partial_t \ell_{022} - 2\kappa_2\ell_{022} + \frac{1}{4\,a}[b_1\ell_{012}-2b_2\ell_{022}+\ell_{12}]^2 - \frac{\varepsilon}{2}[\sigma_1^2\ell_{012}^2+4\rho\sigma_1\sigma_2\ell_{012}\ell_{022}+4\sigma_2^2\ell_{022}^2]\,, \\
0 =&\partial_t \ell_{012} - (\kappa_1+\kappa_2)\ell_{012} + \frac{1}{2\,a}[2b_1\ell_{021}-b_2 l\ell_{012}+\ell_{11}][b_1\ell_{012}-2b_2\ell_{022}+\ell_{12}] \\
&- \frac{\varepsilon}{2}[4\sigma_1^2\ell_{021}\ell_{012}+2\rho\sigma_1\sigma_2(4\ell_{021}\ell_{022}+\ell_{012}^2)+4\sigma_2^2\ell_{012}\ell_{022}]\,, \\
0 =&\partial_t \ell_{021} - 2\kappa_1\ell_{021} + \frac{1}{4\,a}[2b_1\ell_{021}-b_2\ell_{012}+\ell_{11}]^2 - \frac{\varepsilon}{2}[4\sigma_1^2\ell_{021}^2+4\rho\sigma_1\sigma_2\ell_{021}\ell_{012}+\sigma_2^2\ell_{012}^2]\,, \\
0 =&\partial_t \ell_{02} + \kappa_1\theta_1\ell_{012}+2\kappa_2\theta_2\ell_{022}-\kappa_2\ell_{02} + \frac{1}{2\,a}[b_1\ell_{012}-2b_2\ell_{022}+\ell_{12}][b_1\ell_{01}-b_2\ell_{02}+\ell_{10}] \\
&- \frac{\varepsilon}{2}[2\sigma_1^2\ell_{012}\ell_{01}+2\rho\sigma_1\sigma_2(2\ell_{022}\ell_{01}+\ell_{012}\ell_{02})+4\sigma_2^2\ell_{022}\ell_{02}]\,, \\
0 =&\partial_t \ell_{01} + 2\kappa_1\theta_1\ell_{021}+\kappa_2\theta_2\ell_{012}-\kappa_1\ell_{01} + \frac{1}{2\,a}[2b_1\ell_{021}-b_2\ell_{012}+\ell_{11}][b_1\ell_{01}-b_2\ell_{02}+\ell_{10}] \\
&- \frac{\varepsilon}{2}[4\sigma_1^2\ell_{021}l_{01}+2\rho\sigma_1\sigma_2(\ell_{01}\ell_{012}+2\ell_{021}\ell_{02})+2\sigma_2^2\ell_{02}\ell_{012}]\,, \\
0 =&\partial_t \ell_0 + \kappa_1\theta_1\ell_{01}+\kappa_2\theta_2\ell_{02}+(\sigma_1^2\ell_{021}+2\rho\sigma_1\sigma_2\ell_{012}+\sigma_2^2\ell_{022}) \\
&+ \frac{1}{4\,a}[b_1\ell_{01}-b_2\ell_{02}+\ell_{10}]^2 + \sum_{i=1}^{2}\ell_{02i}\int_{-\infty}^{\infty} y^2\,G_i(dy) - \frac{\varepsilon}{2}[\sigma_1^2\ell_{01}^2+2\rho\sigma_1\sigma_2\ell_{01}\ell_{02}+\sigma_2^2\ell_{02}^2]
\end{aligned}
$$

with terminal condition

$$\ell_2(T) = -\alpha, \ell_0(T) = \ell_{01}(T) = \ldots = \ell_{12}(T) = 0\,.$$

■

7.3. Proof of Prop 3

PROOF. Use ansatz (32), and after some tedious computations, equation (31) reduces to the 10-ODEs system:

$$\partial_t \ell_2 + \frac{1}{a}\left[\frac{(b_1(\ell_{11}+1) + b_2(1-\ell_{12})}{2} + \ell_2\right]^2 = 0\,,$$
$$\partial_t \ell_{11} - \kappa_1(\ell_{11}+1) + \frac{1}{a}\left[\frac{(b_1(\ell_{11}+1) + b_2(1-\ell_{12})}{2} + \ell_2\right][2b_1\ell_{021} - b_2\ell_{02} + \ell_{11}] = 0\,,$$
$$\partial_t \ell_{12} - \kappa_2(\ell_{12}-1) + \frac{1}{a}\left[\frac{(b_1(\ell_{11}+1) + b_2(1-\ell_{12})}{2} + \ell_2\right][b_1\ell_{012} - 2b_2\ell_{022} + \ell_{12}] = 0\,,$$
$$\partial_t \ell_{10} + \kappa_1\theta_1(\ell_{11}+1) + \kappa_2\theta_2(\ell_{12}-1) + \frac{1}{a}\left[\frac{(b_1(\ell_{11}+1) + b_2(1-\ell_{12})}{2} + \ell_2\right][b_1\ell_{01} - b_2\ell_{02} + \ell_{10}] = 0\,,$$
$$\partial_t l\ell_{022} - 2\kappa_2\ell_{022} + \frac{1}{4\,a}[b_1\ell_{012} - 2b_2\ell_{022} + \ell_{12}]^2 = 0\,,$$
$$\partial_t \ell_{012} - (\kappa_1+\kappa_2)\ell_{012} + \frac{1}{2\,a}[2b_1\ell_{021} - b_2\ell_{012} + \ell_{11}][b_1\ell_{012} - 2b_2\ell_{022} + \ell_{12}] = 0\,,$$
$$\partial_t \ell_{021} - 2\kappa_1\ell_{021} + \frac{1}{4\,a}[2b_1\ell_{021} - b_2\ell_{012} + \ell_{11}]^2 = 0\,,$$
$$\partial_t \ell_{02} + \kappa_1\theta_1\ell_{012} + 2\kappa_2\theta_2 l\ell_{022} - \kappa_2\ell_{02} + \frac{1}{2\,a}[b_1\ell_{012} - 2b_2\ell_{022} + \ell_{12}][b_1\ell_{01} - b_2\ell_{02} + \ell_{10}] = 0\,,$$
$$\partial_t \ell_{01} + 2\kappa_1\theta_1\ell_{021} + \kappa_2\theta_2\ell_{012} - \kappa_1\ell_{01} + \frac{1}{2\,a}[2b_1\ell_{021} - b_2\ell_{012} + \ell_{11}][b_1\ell_{01} - b_2\ell_{02} + \ell_{10}] = 0\,,$$
$$\partial_t \ell_0 + \kappa_1\theta_1\ell_{01} + \kappa_2\theta_2\ell_{02} + (\sigma_1^2\ell_{021} + 2\rho\sigma_1\sigma_2\ell_{012} + \sigma_2^2\ell_{022}) + \frac{1}{4\,a}[b_1\ell_{01} - b_2\ell_{02} + \ell_{10}]^2 = 0\,,$$

with the terminal condition

$$\ell_2(T) = -\alpha, \ell_0(T) = \ell_{01}(T) = ... = \ell_{12}(T) = 0\,.$$

In the above ODE system, when $b_1 = b_2 = 0$, the ODEs are of Ricatti type and can be solved analytically:

$$
\begin{aligned}
\ell_2(t) &= \frac{a\alpha}{\alpha(t-T)-a}\,,\\
\ell_{11}(t) &= C_1 \exp\left(\kappa_1 t - \frac{1}{a}\int_0^t \ell_2(u)\,du\right) + \frac{\kappa_1 a}{\ell_2(t)-\kappa_1 a}\,,\\
\ell_{12}(t) &= C_2 \exp\left(\kappa_2 t - \frac{1}{a}\int_0^t h_2(u)\,du\right) - \frac{\kappa_2 a}{\ell_2(t)-\kappa_2 a}\,,\\
\ell_{10}(t) &= C_3 \exp\left(-\frac{1}{a}\int_0^t h_2(u)\,du\right) + \int_0^t [-\kappa_1\theta_1(\ell_{11}(u)+1) - \kappa_2\theta_2(\ell_{12}(u)-1)]\, e^{\frac{1}{a}\left(\ell_2(u)-\int_0^u \ell_2(v)\,dv\right)}\,du\,,\\
\ell_{022}(t) &= C_4 e^{2\kappa_2 t} - \frac{1}{4\,a}\int_0^t e^{2\kappa_2(t-u)}\ell_{12}^2(u)\,du\,,\\
\ell_{012}(t) &= C_5 e^{(\kappa_1+\kappa_2)t} - \frac{1}{2\,a}\int_0^t e^{(\kappa_1+\kappa_2)(t-u)}\ell_{12}(u)\ell_{11}(u)\,du\,,\\
\ell_{021}(t) &= C_6 e^{2\kappa_1 t} - \frac{1}{4\,a}\int_0^t e^{2\kappa_1(t-u)}\ell_{11}^2(u)\,du\,,\\
\ell_{02}(t) &= C_7 e^{\kappa_2 t} - \int_0^t e^{\kappa_2(t-u)}\left(\kappa_1\theta_1\ell_{012}(u) + 2\kappa_2\theta_2\ell_{022}(u) + \frac{1}{2\,a}\ell_{12}(u)\ell_{10}(u)\right)\,du\,,\\
\ell_{01}(t) &= C_8 e^{\kappa_1 t} - \int_0^t e^{\kappa_1(t-u)}\left(2\kappa_1\theta_1\ell_{021}(u) + \kappa_2\theta_2\ell_{012}(u) + \frac{1}{2\,a}\ell_{11}(u)\ell_{10}(u)\right)\,du\,,\\
\ell_0(t) &= -\int_0^t \left(\kappa_1\theta_1\ell_{01}(u) + \kappa_2\theta_2\ell_{02}(u) + (\sigma_1^2\ell_{021}(u) + 2\rho\sigma_1\sigma_2\ell_{012}(u) + \sigma_2^2\ell_{022}(u)) + \frac{1}{4\,a}\ell_{10}^2(u)\right)\,du\,,
\end{aligned}
$$

where $C_1, \cdots, C_8$ are constants so the above solutions satisfy their boundary conditions. It is easy to see that the solutions are continuous functions, such that on the finite interval $[0, T]$ they are uniformly bounded. ■

7.4. Proof of Prop 4

PROOF. Use ansatz (34) for (33), we obtain the following ODE system:

$$
\begin{aligned}
\partial_t \ell_2 =& -\frac{1}{a}\left[\frac{(b_1+b_2+b_1\ell_{11}^*-b_2\ell_{12}^*)}{2} + \ell_2^*\right](b_1\ell_{11} - b_2\ell_{12} + 2\ell_2)\\
&+ \frac{A_4}{2}[A_1(\ell_{11}^*+1)^2 - 2\,A_2(\ell_{11}^*+1)(\ell_{12}^*-1) + A_3(\ell_{12}^*-1)^2]\,,\\
\partial_t \ell_{11} =& \kappa_1\ell_{11} - \frac{1}{2\,a}[(2b_1\ell_{021}^* - b_2\ell_{012}^* + \ell_{11}^*)(b_1\ell_{11} - b_2\ell_{12} + 2\ell_2)\\
&+ (b_1(\ell_{11}^*+1) - b_2(\ell_{12}^*-1) + 2\ell_2^*)(2b_1\ell_{021} - b_2\ell_{012} + \ell_{11})]\\
&+ \frac{A_4}{2}[4\,a_1\ell_{021}^*(\ell_{11}^*+1) - 2\,A_2((\ell_{11}^*+1)\ell_{012}^* + 2\ell_{021}^*(\ell_{12}^*-1)) + 2\,A_3(\ell_{12}^*-1)\ell_{012}^*]\,,
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \ell_{12} =& \kappa_2 \ell_{12} - \frac{1}{2\,a}[(b_1\ell^*_{012} - 2b_2\ell^*_{022} + \ell^*_{12})(b_1\ell_{11} - b_2\ell_{12} + 2\ell_2) \\
& + (b_1(\ell^*_{11}+1) - b_2(\ell^*_{12}-1) + 2\ell^*_2)(b_1\ell_{012} - 2b_2\ell_{022} + \ell_{12})] \\
& + \frac{A_4}{2}[2\,A_1(\ell^*_{11}+1)\ell^*_{012} - 2\,A_2(2(\ell^*_{11}+1)\ell^*_{022} + (\ell^*_{12}-1)\ell^*_{012}) + 4\,A_3(\ell^*_{12}-1)\ell^*_{022}]\,, \\
\partial_t \ell_{10} =& -\kappa_1\theta_1\ell_{11} - \kappa_2\theta_2\ell_{12} - \frac{1}{2\,a}[((b_1\ell^*_{01} - b_2\ell^*_{02})(\ell^*_{11}+1) - b_2(\ell^*_{12}-1) + 2\ell^*_2)(b_1\ell_{01} - b_2\ell_{02} + \ell_{10})] \\
& + \frac{A_4}{2}[2\,A_1(\ell^*_{11}+1)\ell^*_{01} - 2\,A_2((\ell^*_{11}+1)\ell^*_{02} + \ell^*_{01}(\ell^*_{12}-1)) + 2\,A_3(\ell^*_{12}-1)\ell^*_{02}]\,, \\
\partial_t \ell_{022} =& 2\kappa_2\ell_{022} - \frac{1}{2\,a}(b_1\ell^*_{012} - 2b_2\ell^*_{022} + \ell^*_{12})(b_1\ell_{012} - 2b_2\ell_{022} + \ell_{12}) \\
& + \frac{A_4}{2}[A_1(\ell^*_{012})^2 - 4\,A_2\ell^*_{012}\ell^*_{022} + 4\,A_3(\ell^*_{022})^2]\,,
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \ell_{012} =& (\kappa_1+\kappa_2)\ell_{012} - \frac{1}{2\,a}[(2b_1\ell^*_{021} - b_2\ell^*_{012} + l\ell^*_{11})(b_1\ell_{012} - 2b_2\ell_{022} + \ell_{12}) \\
& + (b_1\ell^*_{012} - 2b_2\ell^*_{022} + \ell^*_{12})(2b_1\ell_{021} - b_2\ell_{012} + \ell_{11})] \\
& + \frac{A_4}{2}[4\,A_1\ell^*_{021}\ell^*_{012} - 2\,A_2(4\ell^*_{021}\ell^*_{022} + (\ell^*_{012})^2) + 4\,A_3\ell^*_{022}\ell^*_{012}]\,, \\
\partial_t \ell_{021} =& 2\kappa_1\ell_{021} - \frac{1}{2\,a}[(2b_1\ell^*_{021} - b_2\ell^*_{012} + \ell^*_{11})(2b_1\ell_{021} - b_2\ell_{012} + \ell_{11})] \\
& + \frac{A_4}{2}[4\,A_1(\ell^*_{021})^2 - 4\,A_2\ell^*_{021}\ell^*_{012} + A_3(\ell^*_{012})^2]\,,
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \ell_{02} =& -\kappa_1\theta_1\ell_{012} - 2\kappa_2\theta_2\ell_{022} + \kappa_2\ell_{02} - \frac{1}{2\,a}[(b_1\ell^*_{012} - 2b_2\ell^*_{022} + \ell^*_{12})(b_1\ell_{01} - b_2\ell_{01} + \ell_{10}) \\
& + (b_1\ell_{012} - 2b_2\ell_{022} + \ell_{12})(b_1\ell^*_{01} - b_2\ell^*_{02} + \ell^*_{10})] \\
& + \frac{A_4}{2}[2\,A_1\ell^*_{01}\ell^*_{012} - 2\,A_2(2\ell^*_{01}\ell^*_{022} + \ell^*_{012}\ell^*_{02}) + 4\,A_3\ell^*_{02}\ell^*_{022}]\,, \\
\partial_t \ell_{01} =& -2\kappa_1\theta_1\ell_{021} - \kappa_2\theta_2\ell_{012} + \kappa_1\ell_{01} - \frac{1}{2\,a}[(2b_1\ell^*_{021} - b_2\ell^*_{012} + \ell^*_{11})(b_1\ell_{01} - b_2\ell_{02} + \ell_{10}) \\
& + (b_1\ell^*_{01} - b_2\ell^*_{02} + \ell^*_{10})(2b_1\ell_{021} - b_2\ell_{012} + \ell_{11})] \\
& + \frac{A_4}{2}[4\,A_1\ell^*_{01}\ell^*_{021} - 2\,A_2(\ell^*_{01}\ell^*_{012} + 2\ell^*_{021}\ell^*_{02}) + 2\,A_3\ell^*_{02}\ell^*_{012}]\,,
\end{aligned}
$$

where $\ell^*_0, \ell^*_{01}, \cdots, \ell^*_{11}, \ell^*_2$ are the solution of the ODEs in (7.3). Moreover, $l_{01}, \cdots, l_{11}, \ell_2$ satisfy the terminal condition

$$
l_{01}(T) = l_{11}(T) = \cdots = \ell_2(T) = 0\,,
$$

and $A_1, \cdots, A_4$ are the following constants:

$$
A_1 = \sigma_1^2\,, A_2 = \rho\sigma_1\sigma_2\,, A_3 = \sigma_2^2\,, A_4 = \frac{1}{1-\rho^2}\,.
$$

Given $b_1 = b_2 = 0$, for $\ell_2, \ell_{11}, \ell_{12}$, the ODEs have the general form

$$\partial_t F(t) = f(t)F(t) + g(t) \tag{40}$$

where $f(t)$ is the linear combination of some uniform bounded deterministic functions. As when $b_1 = b_2 = 0$, we already established the uniform bounded properties for the solutions of H_0 and ℓ_2^* and all ℓ^* functions are the solution components of H_0. The solution of the ODE (40) is

$$F(t) = Ce^{\int_0^t f(s)\,ds} + \int_0^t g(s)\, e^{\int_0^s f(v)\,dv - f(s)}\, ds$$

where C is a constant, so the solution satisfies the boundary condition. Then we get ℓ_2, l_{11}, l_{12} are bounded.

With the assumption $b_1 = b_2 = 0$, $\ell_2, \ell_{10}, \ell_{11}, \ell_{12}$ and all ℓ^* are bounded, ℓ_{022}, ℓ_{012} and ℓ_{021} have the form

$$\partial_t F(t) = aF(t) + g(t)$$

so by the same argument, $F(t)$ is bounded.

Similarly, after we establish the boundness of $\ell_2, \ell_{10}, \ell_{11}, \ell_{12}$ and $\ell_{022}, \ell_{012}, \ell_{021}$, the ODE ℓ_{01}, ℓ_{02} are both satisfied the same form above, so they are bounded as well. ■

7.5. Proof of Prop 5

PROOF. Use ansatz (36) for (35), and obtain the following ODE system:

$$
\begin{aligned}
\partial_t \ell_2 =& -\frac{1}{a}\left[\frac{(b_1+b_2+b_1\ell_{11}^*-b_2\ell_{12}^*)}{2}+\ell_2^*\right](b_1\ell_{11}-b_2\ell_{12}+2\ell_2)+\frac{\lambda_1}{2}(\ell_{11}^*+1)^2\ell_{12}+\frac{\lambda_2}{2}(\ell_{12}^*-1)^2\ell_{22}\,,\\
\partial_t \ell_{11} =& \kappa_1\ell_{11}-\frac{1}{2\,a}[(2b_1\ell_{021}^*-b_2\ell_{012}^*+\ell_{11}^*)(b_1\ell_{11}-b_2\ell_{12}+2\ell_2)\\
&+(b_1(\ell_{11}^*+1)-b_2(\ell_{12}^*-1)+2\ell_2^*)(2b_1\ell_{021}-b_2\ell_{012}+\ell_{11})]\\
&+2\lambda_1\ell_{021}^*(\ell_{11}^*+1)f_{12}+\lambda_2\ell_{012}^*(\ell_{12}^*-1)f_{22}\,,\\
\partial_t \ell_{12} =& \kappa_2\ell_{12}-\frac{1}{2\,a}[(b_1\ell_{012}^*-2b_2\ell_{022}^*+\ell_{12}^*)(b_1\ell_{11}-b_2\ell_{12}+2\ell_2)\\
&+(b_1(\ell_{11}^*+1)-b_2(\ell_{12}^*-1)+2\ell_2^*)(b_1\ell_{012}-2b_2\ell_{022}+\ell_{12})]\\
&+\lambda_1\ell_{012}^*(\ell_{11}^*+1)f_{12}+2\ell_{022}^*\lambda_2(\ell_{12}^*-1)f_{22}\,,\\
\partial_t \ell_{10} =& -\kappa_1\theta_1\ell_{11}-\kappa_2\theta_2\ell_{12}-\frac{1}{2\,a}[(b_1\ell_{01}^*-b_2\ell_{02}^*+\ell_{10}^*)(b_1\ell_{11}-b_2\ell_{12}+2\ell_2)\\
&+(b_1(\ell_{11}^*+1)-b_2(\ell_{12}^*-1)+2\ell_2^*)(b_1\ell_{01}-b_2\ell_{02}+\ell_{10})]\\
&+\lambda_1[\ell_{01}^*(\ell_{11}^*+1)f_{12}+\ell_{021}^*(\ell_{11}^*+1)f_{13}]+\lambda_2[\ell_{02}^*(\ell_{12}^*-1)f_{22}+\ell_{022}^*(\ell_{12}^*-1)f_{23}]\,,\\
\partial_t \ell_{022} =& 2\kappa_2\ell_{022}-\frac{1}{2\,a}(b_1\ell_{012}^*-2b_2\ell_{022}^*+\ell_{12}^*)(b_1\ell_{012}-2b_2\ell_{022}+\ell_{12})+\frac{\lambda_1}{2}(\ell_{012}^*)^2f_{12}+2\lambda_2(\ell_{022}^*)^2f_{22}\,,\\
\partial_t \ell_{012} =& (\kappa_1+\kappa_2)\ell_{012}-\frac{1}{2\,a}[(2b_1\ell_{021}^*-b_2\ell_{012}^*+\ell_{11}^*)(b_1\ell_{012}-2b_2\ell_{022}+\ell_{12})\\
&+(b_1\ell_{012}^*-2b_2\ell_{022}^*+\ell_{12}^*)(2b_1\ell_{021}-b_2\ell_{012}+\ell_{11})]+2\lambda_1\ell_{021}^*\ell_{012}^*f_{12}+2\lambda_2\ell_{022}^*\ell_{012}^*f_{22}\,,\\
\partial_t \ell_{021} =& 2\kappa_1\ell_{021}-\frac{1}{2\,a}(2b_1\ell_{021}^*-b_2\ell_{012}^*+\ell_{11}^*)(2b_1\ell_{021}-b_2\ell_{012}+\ell_{11})+2\lambda_1(\ell_{021}^*)^2f_{12}+\frac{\lambda_2}{2}(\ell_{012}^*)^2f_{22}\\
\partial_t \ell_{02} =& -\kappa_1\theta_1\ell_{012}-2\kappa_2\theta_2\ell_{022}+\kappa_2\ell_{02}-\frac{1}{2\,a}[(b_1\ell_{012}^*-2b_2\ell_{022}^*+\ell_{12}^*)(b_1\ell_{01}-b_2\ell_{02}+\ell_{10})\\
&+(b_1\ell_{012}-2b_2\ell_{022}+\ell_{12})(b_1\ell_{01}^*-b_2\ell_{02}^*+\ell_{10}^*)]\\
&+\lambda_1(2\ell_{01}^*\ell_{012}^*f_{12}+\ell_{012}^*\ell_{021}^*f_{13})+2\lambda_2(\ell_{02}^*\ell_{022}^*f_{22}+(\ell_{022}^*)^2f_{23})\,,\\
\partial_t \ell_{01} =& -2\kappa_1\theta_1\ell_{021}-\kappa_2\theta_2\ell_{012}+\kappa_1\ell_{01}-\frac{1}{2\,a}[(2b_1\ell_{021}^*-b_2\ell_{012}^*+\ell_{11}^*)(b_1\ell_{01}-b_2\ell_{02}+\ell_{10})\\
&+(b_1\ell_{01}^*-b_2\ell_{02}^*+\ell_{10}^*)(2b_1\ell_{021}-b_2\ell_{012}+\ell_{11})]\\
&+2\lambda_1(\ell_{01}^*\ell_{021}^*f_{12}+(\ell_{021}^*)^2f_{13})+\lambda_2(2\ell_{02}^*\ell_{012}^*f_{22}+\ell_{022}^*\ell_{012}^*f_{23})\,.
\end{aligned}
$$

Here $\ell_0^*, \ell_{01}^*, \cdots, \ell_{11}^*, \ell_2^*$ are the solutions of the ODEs in (7.3). And $\ell_{01}, \cdots, \ell_{11}, \ell_2$ satisfy the terminal condition

$$\ell_{01}(T)=\ell_{11}(T)=\cdots=\ell_2(T)=0\,,$$

and f_{12}, f_{13} are the second and third moments at location 1 and f_{22}, f_{23} are $2nd$ and $3rd$ moments at location 2.

Assuming that the jump components at both locations have finite $2nd$ and $3rd$ moments, and with the permanent impact parameters $b_1 = b_2 = 0$, we can establish the bounded property of H_J iteratively as H_D. ■

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