Some applications of the Ninomiya-Victoir scheme in the context of financial engineering

Christian Bayer

Weierstrass Institute for Applied Analysis and Stochastics

May 21, 2012
Oxford

Outline

Weak approximation of solutions of SDEs

$$dX_t = V_0(X_t)dt + \sum_{i=1}^{d} V_i(X_t) \circ dB_t^i =: \sum_{i=0}^{d} V_i(X_t) \circ dB_t^i, \qquad (1)$$

- $V_0, \ldots, V_d : \mathbb{R}^N \to \mathbb{R}^N$ vector fields
- B_t a d-dimensional Brownian motion, $B_t^0 := t$

Problem

For $f : \mathbb{R}^N \to \mathbb{R}$ sufficiently regular, compute $u(t,x) := E\left[f(X_T) | X_t = x\right]$.

PDE formulation

$\partial_t u + Lu = 0$, where $Lf(x) = V_0 f(x) + \frac{1}{2} \sum_{j=1}^{d} V_i^2 f(x)$, $V_i f(x) := V_i(x) \cdot \nabla f(x)$.

Different approaches

PDE methods: Solve the (linear, second order, parabolic) PDE directly, using finite elements, finite differences,. . . .

Probabilistic methods: Solve the SDE and integrate.

- Discretize SDE to find an approximate solution $\overline{X}_T^{(n)}$.
- Integrate $E\left[f\left(\overline{X}_T^{(n)}\right)\right]$ using (quasi) Monte-Carlo simulation.

Splitting methods: Use structure $L = V_0 + \frac{1}{2}\sum_{j=1}^{d} V_i^2 = \sum_{i=0}^{d} L_i$.

- Solve the PDEs for L_i and combine solutions.
- Probabilistic splitting schemes.

We only consider the probabilistic methods in this talk, with the aim of obtaining higher order methods.

Different approaches

PDE methods: Solve the (linear, second order, parabolic) PDE directly, using finite elements, finite differences,. . . .

Probabilistic methods: Solve the SDE and integrate.

- Discretize SDE to find an approximate solution $\overline{X}_T^{(n)}$.
- Integrate $E\left[f\left(\overline{X}_T^{(n)}\right)\right]$ using (quasi) Monte-Carlo simulation.

Splitting methods: Use structure $L = V_0 + \frac{1}{2}\sum_{j=1}^d V_i^2 = \sum_{i=0}^d L_i$.

- Solve the PDEs for L_i and combine solutions.
- Probabilistic splitting schemes.

We only consider the probabilistic methods in this talk, with the aim of obtaining higher order methods.

Different approaches

PDE methods: Solve the (linear, second order, parabolic) PDE directly, using finite elements, finite differences,. . . .

Probabilistic methods: Solve the SDE and integrate.

- Discretize SDE to find an approximate solution $\overline{X}_T^{(n)}$.
- Integrate $E\left[f\left(\overline{X}_T^{(n)}\right)\right]$ using (quasi) Monte-Carlo simulation.

Splitting methods: Use structure $L = V_0 + \frac{1}{2}\sum_{j=1}^{d} V_i^2 = \sum_{i=0}^{d} L_i$.

- Solve the PDEs for L_i and combine solutions.
- Probabilistic splitting schemes.

We only consider the probabilistic methods in this talk, with the aim of obtaining higher order methods.

Different approaches

PDE methods: Solve the (linear, second order, parabolic) PDE directly, using finite elements, finite differences,. . . .

Probabilistic methods: Solve the SDE and integrate.

- Discretize SDE to find an approximate solution $\overline{X}_T^{(n)}$.
- Integrate $E\left[f\left(\overline{X}_T^{(n)}\right)\right]$ using (quasi) Monte-Carlo simulation.

Splitting methods: Use structure $L = V_0 + \frac{1}{2}\sum_{j=1}^{d} V_i^2 = \sum_{i=0}^{d} L_i$.

- Solve the PDEs for L_i and combine solutions.
- Probabilistic splitting schemes.

We only consider the probabilistic methods in this talk, with the aim of obtaining higher order methods.

Euler discretization of SDEs

- SDE: $dX_t = \sum_{i=0}^{d} V_i(X_t) \circ dB_t^i$
- Naive Euler discretization: $\overline{X}_{t_{j+1}}^{(n)} = \overline{X}_{t_j}^{(n)} + V_0(\overline{X}_{t_j}^{(n)})\Delta t_j + \sum_{i=1}^{d} V_i(\overline{X}_{t_j}^{(n)})\Delta B_j^i$
- Scaling property of Brownian increments: $\Delta B_j^i \sim \mathcal{N}(0, \Delta t_j) \approx \sqrt{\Delta t_j}$, $(\Delta B_j^i)^2 \approx \Delta t_j$
- Correct Euler discretization: $\overline{X}_{t_{j+1}} = \overline{X}_{t_j} + V(\overline{X}_{t_j}^{(n)})\Delta t_j + \sum_{i=1}^{d} V_i(\overline{X}_{t_j}^{(n)})\Delta B_j^i$, with $V(x) = V_0(x) + \frac{1}{2}\sum_{i=1}^{d} DV_i(x) \cdot V_i(x)$

Complications compared to discretization of ODEs

- Higher order terms relevant
- "May not look into future."

Euler discretization of SDEs

- SDE: $dX_t = \sum_{i=0}^{d} V_i(X_t) \circ dB_t^i$
- Naive Euler discretization:
  $\overline{X}_{t_{j+1}}^{(n)} = \overline{X}_{t_j}^{(n)} + V_0(\overline{X}_{t_j}^{(n)})\Delta t_j + \sum_{i=1}^{d} V_i(\overline{X}_{t_j}^{(n)})\Delta B_j^i$
- Scaling property of Brownian increments:
  $\Delta B_j^i \sim \mathcal{N}(0, \Delta t_j) \approx \sqrt{\Delta t_j}$, $(\Delta B_j^i)^2 \approx \Delta t_j$
- Correct Euler discretization:
  $\overline{X}_{t_{j+1}} = \overline{X}_{t_j} + V(\overline{X}_{t_j}^{(n)})\Delta t_j + \sum_{i=1}^{d} V_i(\overline{X}_{t_j}^{(n)})\Delta B_j^i$, with
  $V(x) = V_0(x) + \frac{1}{2}\sum_{i=1}^{d} DV_i(x) \cdot V_i(x)$

Complications compared to discretization of ODEs

- Higher order terms relevant
- "May not look into future."

Euler discretization of SDEs

- SDE: $dX_t = \sum_{i=0}^{d} V_i(X_t) \circ dB_t^i$
- Naive Euler discretization: $\overline{X}_{t_{j+1}}^{(n)} = \overline{X}_{t_j}^{(n)} + V_0(\overline{X}_{t_j}^{(n)})\Delta t_j + \sum_{i=1}^{d} V_i(\overline{X}_{t_j}^{(n)})\Delta B_j^i$
- Scaling property of Brownian increments: $\Delta B_j^i \sim \mathcal{N}(0, \Delta t_j) \approx \sqrt{\Delta t_j}, (\Delta B_j^i)^2 \approx \Delta t_j$
- Correct Euler discretization: $\overline{X}_{t_{j+1}} = \overline{X}_{t_j} + V(\overline{X}_{t_j}^{(n)})\Delta t_j + \sum_{i=1}^{d} V_i(\overline{X}_{t_j}^{(n)})\Delta B_j^i$, with $V(x) = V_0(x) + \frac{1}{2}\sum_{i=1}^{d} DV_i(x) \cdot V_i(x)$

Complications compared to discretization of ODEs

- Higher order terms relevant
- "May not look into future."

Euler discretization of SDEs

- SDE: $dX_t = \sum_{i=0}^{d} V_i(X_t) \circ dB_t^i$
- Naive Euler discretization:
  $\overline{X}_{t_{j+1}}^{(n)} = \overline{X}_{t_j}^{(n)} + V_0(\overline{X}_{t_j}^{(n)})\Delta t_j + \sum_{i=1}^{d} V_i(\overline{X}_{t_j}^{(n)})\Delta B_j^i$
- Scaling property of Brownian increments:
  $\Delta B_j^i \sim \mathcal{N}(0, \Delta t_j) \approx \sqrt{\Delta t_j}$, $(\Delta B_j^i)^2 \approx \Delta t_j$
- Correct Euler discretization:
  $\overline{X}_{t_{j+1}} = \overline{X}_{t_j} + V(\overline{X}_{t_j}^{(n)})\Delta t_j + \sum_{i=1}^{d} V_i(\overline{X}_{t_j}^{(n)})\Delta B_j^i$, with
  $V(x) = V_0(x) + \frac{1}{2}\sum_{i=1}^{d} DV_i(x) \cdot V_i(x)$

Complications compared to discretization of ODEs

- Higher order terms relevant
- "May not look into future."

Integration step

- $\overline{X}_T^{(n)} = \overline{X}_T^{(n)}(\Delta B_1, \dots, \Delta B_n)$.
- Monte Carlo simulation: $\Delta B^{(l)}$ indep. realizations of ΔB,

$$E\left[f\left(\overline{X}_T^{(n)}\right)\right] \approx \frac{1}{M}\sum_{l=1}^{M} f\left(\overline{X}_T^{(n)}(\Delta B_1^{(l)}, \dots, \Delta B_n^{(l)})\right)$$

- Integration error stochastic, but of order $1/\sqrt{M}$, independent of the dimension $n \times d$
- Quasi Monte Carlo simulation: take deterministic vectors $\Delta B^{(l)}$ with special “uniformity” properties
- Integration error of order $1/M$ when dimensions not too high.
- Decomposition of error into discretization error and integration error.

Integration step

- $\overline{X}_T^{(n)} = \overline{X}_T^{(n)}(\Delta B_1, \ldots, \Delta B_n)$.
- Monte Carlo simulation: $\Delta B^{(l)}$ indep. realizations of ΔB,

$$E\left[f\left(\overline{X}_T^{(n)}\right)\right] \approx \frac{1}{M}\sum_{l=1}^{M} f\left(\overline{X}_T^{(n)}(\Delta B_1^{(l)}, \ldots, \Delta B_n^{(l)})\right)$$

- Integration error stochastic, but of order $1/\sqrt{M}$, independent of the dimension $n \times d$
- Quasi Monte Carlo simulation: take deterministic vectors $\Delta B^{(l)}$ with special “uniformity” properties
- Integration error of order $1/M$ when dimensions not too high.
- Decomposition of error into discretization error and integration error.

Integration step

- $\overline{X}_T^{(n)} = \overline{X}_T^{(n)}(\Delta B_1, \ldots, \Delta B_n)$.
- Monte Carlo simulation: $\Delta B^{(l)}$ indep. realizations of ΔB,

$$E\left[f\left(\overline{X}_T^{(n)}\right)\right] \approx \frac{1}{M}\sum_{l=1}^{M} f\left(\overline{X}_T^{(n)}(\Delta B_1^{(l)}, \ldots, \Delta B_n^{(l)})\right)$$

- Integration error stochastic, but of order $1/\sqrt{M}$, independent of the dimension $n \times d$
- Quasi Monte Carlo simulation: take deterministic vectors $\Delta B^{(l)}$ with special “uniformity” properties
- Integration error of order $1/M$ when dimensions not too high.
- Decomposition of error into discretization error and integration error.

Discussion of the probabilistic method

- Order of convergence of Euler scheme: n^{-1} (generically)
- Order of convergence of the (Q)MC simulation: $M^{-1/2}$, M^{-1}
- Integration error dominates.

Goal

Find higher order discretization methods.

- Reduce the dimension $n \times d$ of the integration problem, allowing to rely on quasi Monte Carlo simulation.
- Allows for extremely high precision solvers, which are not available otherwise.
- Geometric solvers.

Discussion of the probabilistic method

- Order of convergence of Euler scheme: n^{-1} (generically)
- Order of convergence of the (Q)MC simulation: $M^{-1/2}$, M^{-1}
- Integration error dominates.

Goal

Find higher order discretization methods.

- Reduce the dimension $n \times d$ of the integration problem, allowing to rely on quasi Monte Carlo simulation.
- Allows for extremely high precision solvers, which are not available otherwise.
- Geometric solvers.

Discussion of the probabilistic method

- Order of convergence of Euler scheme: n^{-1} (generically)
- Order of convergence of the (Q)MC simulation: $M^{-1/2}$, M^{-1}
- Integration error dominates.

Goal

Find higher order discretization methods.

- Reduce the dimension $n \times d$ of the integration problem, allowing to rely on quasi Monte Carlo simulation.
- Allows for extremely high precision solvers, which are not available otherwise.
- Geometric solvers.

Random ODEs

- Let W be a $(d+1)$-dimensional process with paths of bounded variation, define $\widetilde{X}_t = X(W)_t$ by the random ODE

$$\frac{d}{dt}\widetilde{X}_t = \sum_{i=0}^{d} V_i(\widetilde{X}_t)\dot{W}_t^i, \quad \widetilde{X}_0 = x. \tag{2}$$

- Ordinary Taylor expansion:

$$f(\widetilde{X}_t) = \sum_{k=0}^{m} \sum_{(i_1,\dots,i_k)\in\{0,\dots,d\}^k} V_{i_1}\cdots V_{i_k} f(x) W_t^{(i_1,\dots,i_k)} + \widetilde{R}_m(t,x,f)$$

- Stochastic Taylor expansion

$$f(X_t) = \sum_{k=0}^{m} \sum_{(i_1,\dots,i_k)\in\{0,\dots,d\}^k} V_{i_1}\cdots V_{i_k} f(x) B_t^{(i_1,\dots,i_k)} + R_m(t,x,f)$$

- $V_i f(x) := V_i(x)\cdot\nabla f(x)$, $B_t^{(i_1,\dots,i_k)} = \int_0^t B_s^{(i_1,\dots,i_{k-1})} \circ dB_s^{i_k}$.

Cubature on Wiener space

Definition

W is a cubature formula on Wiener space of degree m iff $E\left[W_t^{(i_1,\dots,i_k)}\right] = E\left[B_t^{(i_1,\dots,i_k)}\right]$ for $k \leq m$.

- Cubature formulas with finite support exist (Lyons and Victoir)
- Construction of cubature formulas for $m > 5$ and general d interesting open problem
- Fix a grid $0 = t_0 < t_1 < \cdots < t_n = T$, define W by concatenation of independent cubature formulas (of degree m) on the sub-intervals $[t_i, t_{i+1}]$.
- Global error: $E\left[f(X_T)\right] - E\left[f(\widetilde{X}_T^{(n)})\right] = O((\max_j \Delta t_j)^{(m-1)/2})$

Cubature on Wiener space

Definition

W is a cubature formula on Wiener space of degree m iff $E\left[W_t^{(i_1,\dots,i_k)}\right] = E\left[B_t^{(i_1,\dots,i_k)}\right]$ for $k \leq m$.

- Cubature formulas with finite support exist (Lyons and Victoir)
- Construction of cubature formulas for $m > 5$ and general d interesting open problem
- Fix a grid $0 = t_0 < t_1 < \cdots < t_n = T$, define W by concatenation of independent cubature formulas (of degree m) on the sub-intervals $[t_i, t_{i+1}]$.
- Global error: $E\left[f(X_T)\right] - E\left[f(\widetilde{X}_T^{(n)})\right] = \mathcal{O}((\max_j \Delta t_j)^{(m-1)/2})$

Abstract splitting

$$E[f(X_t)|X_0 = x] =: P_t f(x) = \exp\left(t\left(V_0 + \frac{1}{2}\sum_{i=1}^{d} V_i^2\right)\right) f(x)$$

- General splitting: $V_0 + \frac{1}{2}\sum_{i=1}^{d} V_i^2 = \sum_{i=0}^{d} U_i$, then approximate $P_t \approx \prod_j e^{t\gamma_j U_{i(j)}}$
- Maximal order of convergence: 2 for positive weights γ

Example

- $Q_t = e^{tU_0} \cdots e^{tU_d}$, $Q_t^* = e^{tU_d} \cdots e^{tU_0}$ (Lie-Trotter splitting or symplectic Euler method)
- $Q_t = \frac{1}{2}(e^{tU_0} \cdots e^{tU_d} + e^{tU_d} \cdots e^{tU_0})$ (symmetrically weighted sequential splitting)

The Ninomiya-Victoir method

- On a (uniform) grid $0 = t_0 < \cdots < t_n = T$ set $\Delta t_i := t_{i+1} - t_i$, $\Delta B_i^j := B_{t_{i+1}}^j - B_{t_i}^j$, Λ_i Bernoulli-distributed
- Set $\overline{X}_0 = x$ and iteratively

$$\overline{X}_{i+1} := \begin{cases} e^{\frac{\Delta t_i}{2} V_0} e^{\Delta B_i^d V_d} \cdots e^{\Delta B_i^1 V_1} e^{\frac{\Delta t_i}{2} V_0} \overline{X}_i, & \Lambda_i = 1, \\ e^{\frac{\Delta t_i}{2} V_0} e^{\Delta B_i^1 V_1} \cdots e^{\Delta B_i^d V_d} e^{\frac{\Delta t_i}{2} V_0} \overline{X}_i, & \Lambda_i = -1. \end{cases} \quad (3)$$

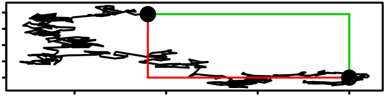
- $e^{sV_i} x := z(1)$, where $\dot{z}(t) = sV_i(z(t))$, $z(0) = x$
- Global error: $E[f(X_T)] - E\left[f\left(\overline{X}_n\right)\right] = \mathcal{O}((\sup \Delta t_i)^2)$
- Interpretation as cubature method and splitting method

The Ninomiya-Victoir method

- On a (uniform) grid $0 = t_0 < \cdots < t_n = T$ set $\Delta t_i := t_{i+1} - t_i$, $\Delta B^j_i := B^j_{t_{i+1}} - B^j_{t_i}$, Λ_i Bernoulli-distributed
- Set $\overline{X}_0 = x$ and iteratively

$$\overline{X}_{i+1} := \begin{cases} e^{\frac{\Delta t_i}{2} V_0} e^{\Delta B^d_i V_d} \cdots e^{\Delta B^1_i V_1} e^{\frac{\Delta t_i}{2} V_0} \overline{X}_i, & \Lambda_i = 1, \\ e^{\frac{\Delta t_i}{2} V_0} e^{\Delta B^1_i V_1} \cdots e^{\Delta B^d_i V_d} e^{\frac{\Delta t_i}{2} V_0} \overline{X}_i, & \Lambda_i = -1. \end{cases} \tag{3}$$

- $e^{sV_i} x := z(1)$, where $\dot{z}(t) = sV_i(z(t))$, $z(0) = x$
- Global error: $E\left[f(X_T)\right] - E\left[f\left(\overline{X}_n\right)\right] = \mathcal{O}((\sup \Delta t_i)^2)$
- Interpretation as cubature method and splitting method



The Ninomiya-Victoir method

- On a (uniform) grid $0 = t_0 < \cdots < t_n = T$ set $\Delta t_i := t_{i+1} - t_i$, $\Delta B_i^j := B_{t_{i+1}}^j - B_{t_i}^j$, Λ_i Bernoulli-distributed
- Set $\overline{X}_0 = x$ and iteratively

$$\overline{X}_{i+1} := \begin{cases} e^{\frac{\Delta t_i}{2} V_0} e^{\Delta B_i^d V_d} \cdots e^{\Delta B_i^1 V_1} e^{\frac{\Delta t_i}{2} V_0} \overline{X}_i, & \Lambda_i = 1, \\ e^{\frac{\Delta t_i}{2} V_0} e^{\Delta B_i^1 V_1} \cdots e^{\Delta B_i^d V_d} e^{\frac{\Delta t_i}{2} V_0} \overline{X}_i, & \Lambda_i = -1. \end{cases} \tag{3}$$

- $e^{sV_i} x := z(1)$, where $\dot{z}(t) = sV_i(z(t))$, $z(0) = x$
- Global error: $E\left[f(X_T)\right] - E\left[f\left(\overline{X}_n\right)\right] = \mathcal{O}((\sup \Delta t_i)^2)$
- Interpretation as cubature method and splitting method

$$Q_{\Delta t}^{NV} = \frac{1}{2} e^{\frac{\Delta t}{2} L_0} e^{\Delta t L_1} \cdots e^{\Delta t L_d} e^{\frac{\Delta t}{2} L_0} + \frac{1}{2} e^{\frac{\Delta t}{2} L_0} e^{\Delta t L_d} \cdots e^{\Delta t L_1} e^{\frac{\Delta t}{2} L_0},$$

where $L_0 f(x) = V_0 f(x)$, $L_i f(x) = \frac{1}{2} V_i^2 f(x)$,
$Q_{\Delta t}^{NV} \approx P_{\Delta t} := e^{\Delta t L_0 + \Delta t \sum_{i=1}^d L_i}$

ODEs for Ninomiya-Victoir

- Requires $\exp(sV_0), \exp(sV_1), \ldots, \exp(sV_d)$
- Numerical solution of ODEs possible, see Ninomiya and Ninomiya.
- Experience suggests that explicit solutions preferable whenever available.
- Question: Which relevant models in mathematical finance allow for explicit formulas of all required terms $\exp(sV_0), \exp(sV_1), \ldots, \exp(sV_d)$?
- Diffusion vector-fields $V_1, \ldots, V_d$ often simple enough, Stratonovich correction causing problems,

$$V_0(x) = V(x) - \frac{1}{2} \sum_{i=1}^{d} DV_i(x) \cdot V_i(x).$$

- Idea: move correction terms back to diffusion part.

ODEs for Ninomiya-Victoir

- Requires $\exp(sV_0), \exp(sV_1), \ldots, \exp(sV_d)$
- Numerical solution of ODEs possible, see Ninomiya and Ninomiya.
- Experience suggests that explicit solutions preferable whenever available.
- Question: Which relevant models in mathematical finance allow for explicit formulas of all required terms $\exp(sV_0), \exp(sV_1), \ldots, \exp(sV_d)$?
- Diffusion vector-fields $V_1, \ldots, V_d$ often simple enough, Stratonovich correction causing problems,

$$V_0(x) = V(x) - \frac{1}{2}\sum_{i=1}^{d} DV_i(x) \cdot V_i(x).$$

- Idea: move correction terms back to diffusion part.

ODEs for Ninomiya-Victoir

- Requires $\exp(sV_0), \exp(sV_1), \ldots, \exp(sV_d)$
- Numerical solution of ODEs possible, see Ninomiya and Ninomiya.
- Experience suggests that explicit solutions preferable whenever available.
- Question: Which relevant models in mathematical finance allow for explicit formulas of all required terms $\exp(sV_0), \exp(sV_1), \ldots, \exp(sV_d)$?
- Diffusion vector-fields $V_1, \ldots, V_d$ often simple enough, Stratonovich correction causing problems,

$$V_0(x) = V(x) - \frac{1}{2} \sum_{i=1}^{d} DV_i(x) \cdot V_i(x).$$

- Idea: move correction terms back to diffusion part.

Drift trick

Reformulation

$$dX_t = V_0^{(\gamma)}(X_t)dt + \sum_{i=1}^{d} V_i(X_t) \circ d(B_t^i + \gamma^i t),$$

where $V_0^{(\gamma)}(x) := V_0(x) - \sum_{i=1}^d V_i(x)\gamma^i$.

- Use Ninomiya-Victoir with V_0 replaced by $V_0^{(\gamma)}$ and ΔB^i replaced by $\Delta B^i + \gamma^i \Delta t$.
- Second order convergence retained.
- Cubature method also obvious.
- Non-standard splitting:
  $L = V_0 + \frac{1}{2}\sum_{i=1}^d V_i^2 = V_0^{(\gamma)} + \sum_{i=1}^d \left(\frac{1}{2}V_i^2 + \gamma^i V_i\right)$

Girsanov transform

- Let $\mathcal{E}_t := \exp\left(\langle\gamma,\, B_t\rangle - \frac{1}{2}\|\gamma\|^2\, t\right)$ and Q be defined by $\frac{dQ}{dP} = \mathcal{E}_T$.
- We have

$$E_P[f(X_T)] = E_Q[f(Y_T)] = E_P[f(Y_T)\mathcal{E}_T],$$

  where Y_T solves the SDE with $V_0^{(\gamma)}, V_1, \ldots, V_d$.
- But: $\mathrm{Var}[\mathcal{E}_T] = e^{\|\gamma\|^2 T} - 1$.

Generalized SABR model

Model

$$dX_t^1 = a\left(X_t^2\right)^\alpha \left(X_t^1\right)^\beta dB_t^1,$$
$$dX_t^2 = \kappa(\theta - X_t^2)dt + bX_t^2(\rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2),$$

where $1/2 \le \alpha, \beta \le 1$. (SABR: $\alpha = 1, \kappa = 0$.)

$$e^{sV_1}x = \begin{pmatrix} g_1(s,x) \\ x^2 e^{b\rho s} \end{pmatrix}, \quad e^{sV_2}x = \begin{pmatrix} x^1 \\ x^2 e^{b\sqrt{1-\rho^2}s} \end{pmatrix},$$

$$g_1(s,x) = \begin{cases} \left[(1-\beta)\frac{a(x^2)^\alpha}{\alpha b \rho}\left(e^{\alpha b \rho s} - 1\right) + \left(x^1\right)^{1-\beta}\right]_+^{1/(1-\beta)}, & \beta < 1, \\ x^1 \exp\left(\frac{a(x^2)^\alpha}{\alpha b \rho}\left(e^{\alpha b \rho s} - 1\right)\right), & \beta = 1. \end{cases}$$

Generalized SABR model – 2

- No explicit formula for $e^{sV_0}x$, where

$$V_0(x) = \begin{pmatrix} -\frac{1}{2}a^2\beta\left(x^2\right)^{2\alpha}\left(x^1\right)^{2\beta-1} - \frac{1}{2}\alpha ab\rho\left(x^2\right)^{\alpha}\left(x^1\right)^{\beta} \\ \kappa\theta - \left(\kappa + \frac{1}{2}b^2\right)x^2 \end{pmatrix}$$

- Drift trick: choose $\gamma \in \mathbb{R}^d$, set $V_0^{(\gamma)}(x) := V_0(x) - \sum_{i=1}^d \gamma^i V_i(x)$ and consider

$$dX_t = V_0^{(\gamma)}(X_t)dt + \sum_{i=1}^d V_i(X_t) \circ d\left(B_t^i + \gamma^i t\right)$$

Generalized SABR model – 2

- No explicit formula for $e^{sV_0}x$, where

$$V_0(x) = \begin{pmatrix} -\frac{1}{2}a^2\beta\left(x^2\right)^{2\alpha}\left(x^1\right)^{2\beta-1} - \frac{1}{2}\alpha ab\rho\left(x^2\right)^{\alpha}\left(x^1\right)^{\beta} \\ \kappa\theta - \left(\kappa + \frac{1}{2}b^2\right)x^2 \end{pmatrix}$$

- Drift trick: choose $\gamma \in \mathbb{R}^d$, set $V_0^{(\gamma)}(x) := V_0(x) - \sum_{i=1}^d \gamma^i V_i(x)$ and consider

$$dX_t = V_0^{(\gamma)}(X_t)dt + \sum_{i=1}^{d} V_i(X_t) \circ d\left(B_t^i + \gamma^i t\right)$$

Generalized SABR model – 3

- Choose $\gamma^1 = -\frac{1}{2}\alpha b \rho$, $\gamma^2 = \frac{\alpha b \rho^2 - 2\kappa/b - b}{2\sqrt{1-\rho^2}}$ to obtain

$$V_0^{(\gamma)}(x) = \begin{pmatrix} -\frac{1}{2}a^2\beta\left(x^2\right)^{2\alpha}\left(x^1\right)^{2\beta-1} \\ \kappa\theta \end{pmatrix}$$

- Explicit solution: $e^{sV_0^{(\gamma)}}x = \left(g_0(s,x), \kappa\theta s + x^2\right)$, with

$$g_0(s,x) = \begin{cases} \left[-\theta^2\beta(1-\beta)P(s,x) + \left(x^1\right)^{2(1-\beta)}\right]_+^{1/2(1-\beta)}, & \beta < 1, \\ x^1 \exp\left(-\frac{1}{2}a^2 P(s,x)\right), & \beta = 1, \end{cases}$$

$$P(s,x) = \frac{1}{(2\alpha+1)\kappa\theta}\left(\left(\kappa\theta s + x^2\right)^{2\alpha+1} - \left(x^2\right)^{2\alpha+1}\right)$$

Generalized SABR model – 3

- Choose $\gamma^1 = -\frac{1}{2}\alpha b \rho$, $\gamma^2 = \frac{\alpha b \rho^2 - 2\kappa/b - b}{2\sqrt{1-\rho^2}}$ to obtain

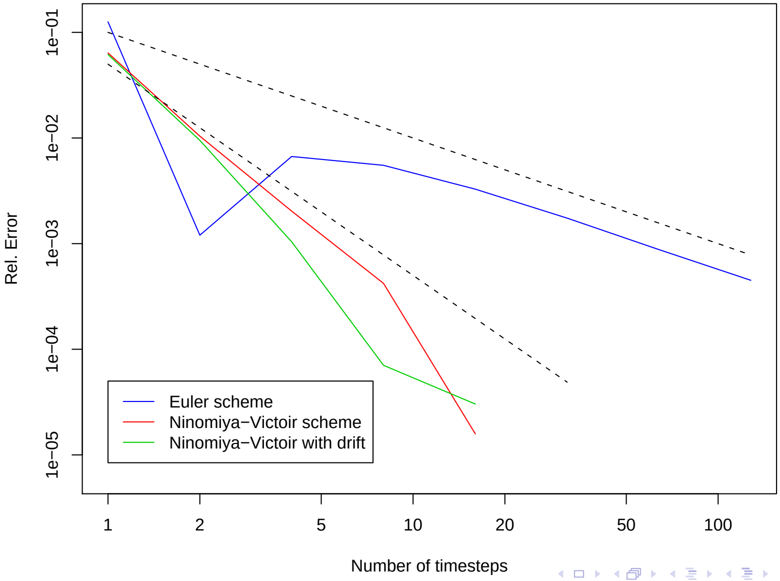
$$V_0^{(\gamma)}(x) = \begin{pmatrix} -\frac{1}{2}a^2\beta\left(x^2\right)^{2\alpha}\left(x^1\right)^{2\beta-1} \\ \kappa\theta \end{pmatrix}$$

- Explicit solution: $e^{sV_0^{(\gamma)}}x = \left(g_0(s,x), \kappa\theta s + x^2\right)$, with

$$g_0(s,x) = \begin{cases} \left[-\theta^2\beta(1-\beta)P(s,x) + \left(x^1\right)^{2(1-\beta)}\right]_+^{1/2(1-\beta)}, & \beta < 1, \\ x^1 \exp\left(-\frac{1}{2}a^2 P(s,x)\right), & \beta = 1, \end{cases}$$

$$P(s,x) = \frac{1}{(2\alpha+1)\kappa\theta}\left(\left(\kappa\theta s + x^2\right)^{2\alpha+1} - \left(x^2\right)^{2\alpha+1}\right)$$

Generalized SABR – Numerical experiment



Generalized SABR – Computational time

Method	n	M	Rel. Error	Time
Euler	32	8192000	0.00174	91.94 sec
Ninomiya-Victoir	4	2048000	0.00204	13.93 sec
NV with drift	4	1024000	0.00104	2.88 sec

Multi-dimensional generalized SABR

Model

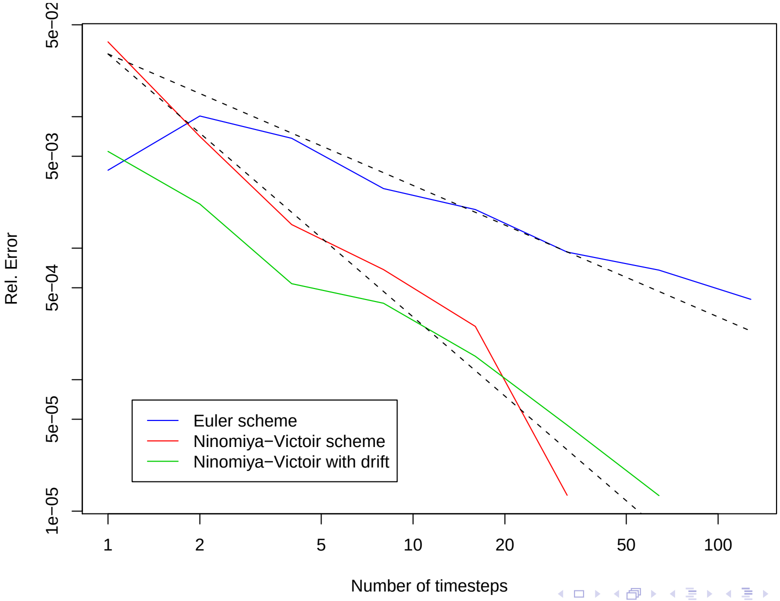
$$\mathrm{d}X_i(t) = a_i Y_i(t)^{\alpha_i} X_i(t)^{\beta_i} \mathrm{d}\widetilde{B}^i_t$$
$$\mathrm{d}Y_i(t) = \kappa_i(\theta_i - Y_i(t))\mathrm{d}t + b_i Y_i(t)\mathrm{d}\widetilde{W}^i_t$$

$\widetilde{B}$ and $\widetilde{W}$ correlated Brownian motions.

- Drift trick allows solving all ODEs explicitly provided that the correlation matrix has full rank.
- Here we use 4 assets, i.e., dimension $N = 8$, $d = 8$.

Method	n	M	Rel. Error	Time
Euler	32	2048000	0.000934	246.65 sec
Ninomiya-Victoir	4	1024000	0.002017	52.33 sec
NV with drift	4	1024000	0.000862	35.31 sec

Multi-dimensional Generalized SABR



Heat kernel expansion

- Consider the linear, parabolic PDE $u_t + \frac{1}{2}\sum_{i,j=1}^{n} a^{i,j} u_{x_i,x_j} + \sum_{i=1}^{n} b^i u_{x_i} = 0.$
- Change geometry such that PDE is transformed to heat equation $u_t + \frac{1}{2}\Delta_B u + V \cdot \nabla u = 0.$
- Use asymptotic formula for fundamental solution (transition density).
- Riemannian metric $g^{i,j} = a^{i,j}$ induces a Riemannian (geodesic) distance d and a Laplace-Beltrami operator Δ_B.
- Heat kernel expansion:

$$p(x,y,T) = \sqrt{\det g_{i,j}(y)}\, U_k(x,y,T) \frac{1}{(2\pi T)^{n/2}} e^{-\frac{d(x,y)^2}{2T}} + \mathcal{O}(T^{k+1}).$$

- $U_k = \sum_{i=0}^{k} u_i T^i$, coefficients u_0 and u_1 have geometric interpretations.

Heat kernel expansion

- Consider the linear, parabolic PDE $u_t + \frac{1}{2}\sum_{i,j=1}^{n} a^{i,j} u_{x_i,x_j} + \sum_{i=1}^{n} b^i u_{x_i} = 0.$
- Change geometry such that PDE is transformed to heat equation $u_t + \frac{1}{2}\Delta_B u + V \cdot \nabla u = 0.$
- Use asymptotic formula for fundamental solution (transition density).
- Riemannian metric $g^{i,j} = a^{i,j}$ induces a Riemannian (geodesic) distance d and a Laplace-Beltrami operator Δ_B.
- Heat kernel expansion:

$$p(x,y,T) = \sqrt{\det g_{i,j}(y)}\, U_k(x,y,T) \frac{1}{(2\pi T)^{n/2}} e^{-\frac{d(x,y)^2}{2T}} + \mathcal{O}(T^{k+1}).$$

- $U_k = \sum_{i=0}^{k} u_i T^i$, coefficients u_0 and u_1 have geometric interpretations.

Applications to basket options

- Consider a basket of n stocks given by CEV processes

$$dS_t^i = \sigma_i (S_t^i)^{\beta_i} dW_t^i, \ i = 1, \dots, n,$$

where $\mathrm{cor}(W_t^i, W_t^j) = \rho_{i,j}$.

- Goal: price a basket (spread) option with payoff $\left(\sum_{i=1}^n w_i S_T^i - K\right)^+$.
- $$\int_{\mathbb{R}^n} \left(\sum_{i=1}^n w_i x_i - K\right)^+ p(S_0, x, T) dx$$
  - Replace transition density p by its 0-order expansion.
  - Compute integral by saddle-point approximation.
- Implied Black-Scholes (Bachelier) volatilities $\sigma_0 + T\sigma_1$ obtained by comparison of coefficients.

Applications to basket options

- Consider a basket of n stocks given by CEV processes

$$dS_t^i = \sigma_i (S_t^i)^{\beta_i} dW_t^i, \; i = 1, \dots, n,$$

  where $\mathrm{cor}(W_t^i, W_t^j) = \rho_{i,j}$.
- Goal: price a basket (spread) option with payoff $\left(\sum_{i=1}^n w_i S_T^i - K\right)^+$.
- $$\int_{\mathbb{R}^n} \left(\sum_{i=1}^n w_i x_i - K \right)^+ p(S_0, x, T) dx$$
  - Replace transition density p by its 0-order expansion.
  - Compute integral by saddle-point approximation.
- Implied Black-Scholes (Bachelier) volatilities $\sigma_0 + T\sigma_1$ obtained by comparison of coefficients.

Implementation

- Obtain closed form results up to solving a quadratic optimization problem in $\mathbb{R}^n$ with non-linear constraints.
- Validate the approximation formula by comparison with values obtained by the Ninomiya-Victoir method
  - Variance reduction by the Mean value Monte Carlo method
  - integration using Sobol numbers

T	$K = 27.9$	$K = 28.5$	$K = 29.1$	$K = 29.7$
0.5	-4.8×10^{-5}	-3.5×10^{-5}	-3.8×10^{-5}	-5.4×10^{-5}
1	1.6×10^{-4}	1.9×10^{-4}	2.3×10^{-4}	2.6×10^{-4}
5	7.4×10^{-3}	7.8×10^{-3}	8.3×10^{-3}	8.7×10^{-3}

Table: Relative error of the asymptotic price formula for a 10-dim. basket option; option is at the money for $K = 29$.

The DMR model

- Consider Gatheral's double mean reverting stochastic volatility model (Bühler's affine variance curve model)

$$\begin{cases} dS_t = \sqrt{v_t} S_t dW_t^1, \\ dv_t = \kappa_1 (v_t' - v_t) dt + \xi_1 v_t^{\alpha_1} dW_t^2, \\ dv_t' = \kappa_2 (\theta - v_t') dt + \xi_2 (v_t')^{\alpha_2} dW_t^3, \end{cases}$$

where $\mathrm{cor}(W_t^i, W_t^j) = \rho_{i,j}$.

- Goal of the model: price models on both SPX and VIX consistently.
- Variance v_t mean-reverts to a value, which moves slowly over time.
- Calibration to both SPX and VIX options.

Calibration – 1

- Constant parameters $\kappa_1, \kappa_2, \theta, \rho_{2,3}$ and α_1, α_2.
- Closed form expression for variance swaps

$$E\left[\int_t^{t+\tau} v_s ds \middle| \mathcal{F}_t\right] = \theta\tau + (v_t - \theta)\frac{1 - e^{-\kappa_1\tau}}{\kappa_1} + \\ + (v'_t - \theta)\frac{\kappa_1}{\kappa_1 - \kappa_2}\left(\frac{1 - e^{-\kappa_1\tau}}{\kappa_1} - \frac{1 - e^{-\kappa_2\tau}}{\kappa_2}\right)$$

- Allows direct estimation of $\kappa_1, \kappa_2, \theta$ from historical variance swap / VIX data and construction of v_t, v'_t time series.
- $\rho_{2,3}$ historical correlation.
- α_1, α_2 estimated from a parametric ansatz (SABR formula) from VIX.

Calibration – 2

- Parameters ξ_1, ξ_2 and $\rho_{1,2}, \rho_{1,3}$ left for daily calibration
- ξ_1, ξ_2 calibrated to VIX options with theoretical prices computed using the Ninomiya-Victoir method
- Explicit solutions to ODEs available by additional drift split $V_0 = V_{0,1} + V_{0,2}$
- Speed-up of factor 10 as compared to Euler method, 20 time-steps instead of 1000
- $\rho_{1,2}, \rho_{1,3}$ calibrated to SPX options using Euler with 30 time-steps – even for Ninomiya-Victoir we would have to use at least 14 time-steps corresponding to 14 different maturities.
- Calibration can be done in 30 seconds.

References

- Bayer, C., Friz, P., Loeffen, R. *Semi-closed form cubature and applications to financial diffusion models*, preprint, 2011.
- Gatheral, J. *Consistent modelling of SPX and VIX option*, presentation at the Bachelier conference, 2008.
- Gatheral, J., Hsu, E., Laurence, P., Ouyang, C., Wang, T. *Asymptotics of implied volatility in local volatility models*, Math. Fin., 2010.
- Lyons, T., Victoir, N. *Cubature on Wiener space*, Proc. R. Soc. Lond. Ser. A, 460(2041), 2004.
- Ninomiya, S., Victoir, N. *Weak approximation of SDEs and appl. to derivative pricing*, Appl. Math. Fin., 15(1-2), 2008.
- Oshima, K., Teichmann, J., Veluščček, D. *A new extrapolation method for weak approximation schemes with applications*, preprint, 2011.

Construction of splitting and cubature schemes

- Gaussian K schemes: for an m-order approximation Q_t of P_t, find a random variable $Z_{t,x,f}$ s.t. $E[Z] = Q_t f(x)$.
- Approximate $\exp\left(t\left(v_0 + \frac{1}{2}\sum_{i=1}^d v_i^2\right)\right) \approx E[e^Y]$ for Y taking values in the (step-m nilpotent) free Lie algebra generated by $v_0, \ldots, v_d$.
- Construction of Y comparable to construction of classical cubature formulas on $\mathbb{R}^d$.
- Link to cubature on Wiener space: $\exp\left(t\left(v_0 + \frac{1}{2}\sum_{i=1}^d v_i^2\right)\right)$ can be interpreted as expectation of the random variable $(B_t^{(i_1,\ldots,i_k)})_{k\leq m}$.

Construction of splitting and cubature schemes

- Gaussian K schemes: for an m-order approximation Q_t of P_t, find a random variable $Z_{t,x,f}$ s.t. $E[Z] = Q_t f(x)$.
- Approximate $\exp\left(t\left(v_0 + \frac{1}{2}\sum_{i=1}^{d} v_i^2\right)\right) \approx E[e^Y]$ for Y taking values in the (step-m nilpotent) free Lie algebra generated by $v_0, \ldots, v_d$.
- Construction of Y comparable to construction of classical cubature formulas on $\mathbb{R}^d$.
- Link to cubature on Wiener space: $\exp\left(t\left(v_0 + \frac{1}{2}\sum_{i=1}^{d} v_i^2\right)\right)$ can be interpreted as expectation of the random variable $(B_t^{(i_1,\ldots,i_k)})_{k \le m}$.

Iterating the scheme

- Given $\|P_t - Q_t\| \leq t^{\ell+1}$ obtained by cubature, splitting, ...
- Time grid: $0 = t_0 < t_1 < \cdots < t_N = T$, $Q_T^{(N)} := Q_{\Delta t_N} \cdots Q_{\Delta t_1}$.

$$\left\| P_T f - Q_T^{(N)} f \right\|_\infty \leq \sum_{k=1}^{N} \left\| Q_{\Delta t_N} \cdots Q_{\Delta t_{k+1}} P_{t_k} f - Q_{\Delta t_N} \cdots Q_{\Delta t_k} P_{t_{k-1}} f \right\|_\infty$$

$$\leq \sum_{k=1}^{N} \left\| Q_{\Delta t_N} \cdots Q_{\Delta t_{k+1}} \right\| \left\| (P_{\Delta t_k} - Q_{\Delta t_k}) P_{t_{k-1}} f \right\|_\infty$$

$$\leq \text{const} \sum_{k=1}^{N} \Delta t_k^{l+1} \leq \text{const} \left(\max_k \Delta t_k \right)^\ell .$$

- Relax regularity assumptions under Hörmander type conditions.

Iterating the scheme

- Given $\|P_t - Q_t\| \leq t^{\ell+1}$ obtained by cubature, splitting, ...
- Time grid: $0 = t_0 < t_1 < \cdots < t_N = T$, $Q_T^{(N)} := Q_{\Delta t_N} \cdots Q_{\Delta t_1}$.

$$\left\|P_T f - Q_T^{(N)} f\right\|_\infty \leq \sum_{k=1}^{N} \left\|Q_{\Delta t_N} \cdots Q_{\Delta t_{k+1}} P_{t_k} f - Q_{\Delta t_N} \cdots Q_{\Delta t_k} P_{t_{k-1}} f\right\|_\infty$$

$$\leq \sum_{k=1}^{N} \left\|Q_{\Delta t_N} \cdots Q_{\Delta t_{k+1}}\right\| \left\|(P_{\Delta t_k} - Q_{\Delta t_k}) P_{t_{k-1}} f\right\|_\infty$$

$$\leq \mathrm{const} \sum_{k=1}^{N} \Delta t_k^{l+1} \leq \mathrm{const} \left(\max_k \Delta t_k\right)^{\ell}.$$

- Relax regularity assumptions under Hörmander type conditions.