On the number of components of random nodal sets

work in progress with Fedor Nazarov (Kent)

Oxford Stochastic Analysis Seminar

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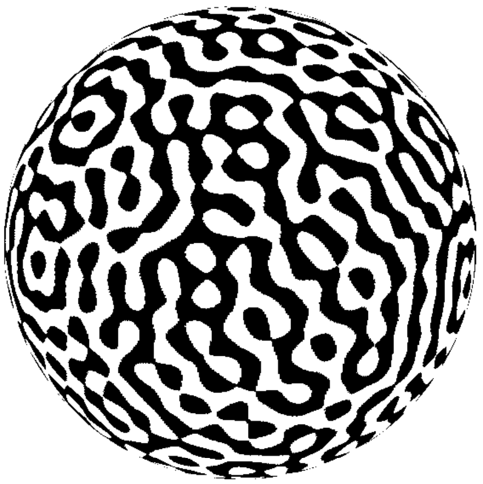
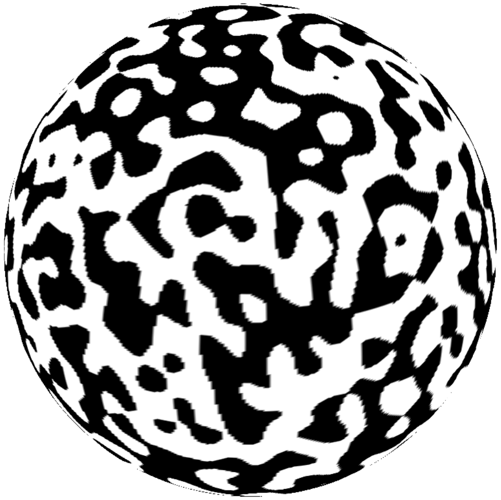
Nodal protraits created by Alex Barnett

Gaussian spherical harmonic of degree 40



Gaussian linear combination of spherical harmonic of degrees ≤ 40



Part I: Zero sets of translation-invariant Gaussian functions

$F\colon \mathbb{R}^m \to \mathbb{R}$ a random Gaussian function with translation-invariant distribution;

$\mathcal{E}\{F(u)F(v)\} = k(u-v)$, $k = \widehat{\rho}$ (the Fourier transform)

$\rho \in M^+(\mathbb{R}^m)$, symmetric w.r.t. origin, the spectral measure of F.

<u>We assume</u>: for some $p > 4$, $\displaystyle\int_{\mathbb{R}^m} |\lambda|^p \, \mathrm{d}\rho(\lambda) < \infty$

$\Longrightarrow$ a.s. the random function F is C^2-smooth

$N(R;F)$ the number of connected components of the zero set $Z(F)$ that are contained in the open ball $\big\{|x| < R\big\}$

We are interested in the asymptotic behaviour of r.v. $N(R;F)$ as $R \to \infty$

THEOREM I: *Suppose that the measure ρ has no atoms and is not supported by a hyperplane.*

(i) *There exists a constant $\nu(\rho) \geq 0$ s.t.*

$$\lim_{R\to\infty} \frac{\mathcal{E}N(R;F)}{\mathrm{vol}B(R)} = \nu(\rho) \quad \text{and} \quad \lim_{R\to\infty} \frac{N(R;F)}{\mathrm{vol}B(R)} = \nu(\rho) \quad \text{a.s.}\,.$$

(ii) *The limiting constant $\nu(\rho)$ is* <u>*positive*</u> *provided that*

(∗) $\exists$ *a compactly supported symmetric measure μ with $\mathrm{spt}(\mu) \subset \mathrm{spt}(\rho)$ s.t. the zero set of its Fourier transform $\widehat{\mu}$ has a bounded component.*

(iii) *If (∗) does not hold then a.s. $Z(F)$ has no bounded connected components.*

How to check condition $(*)$?

$\exists$ a compactly supported symmetric measure μ with $\mathrm{spt}(\mu) \subset \mathrm{spt}(\rho)$ s.t. the zero set of its Fourier transform $\widehat{\mu}$ has a bounded component.

A simple and crude sufficient condition:

- $\mathrm{spt}(\rho)$ *contains a sphere centered at the origin.*

Proof: take $\mu =$ the Lebesgue measure on that sphere $\Longrightarrow$ $\widehat{\mu}$ is radially symmetric, vanishes on concentric spheres with radii tending to ∞ $\square$

Using a little bit of harmonic analysis, one can go further:

- $\mathrm{spt}(\rho)$ *contains an open subset of a sphere centered at the origin*

In the planar case $(m = 2)$, there is even a more simple sufficient condition:

- $\mathrm{spt}(\rho)$ *contains a compact set that cannot be covered by finitely many segments*

Proof: Take $\lambda_1, \lambda_2 \in \mathrm{spt}(\rho)$, $\lambda_2 \neq c\lambda_1$, and consider

$$\cos\big(\lambda_1 \cdot x\big) + \cos\big(\lambda_2 \cdot x\big) = 2\cos\Big(\frac{\lambda_1 + \lambda_2}{2} \cdot x\Big) \cos\Big(\frac{\lambda_1 - \lambda_2}{2} \cdot x\Big)$$

Then add a (carefully chosen) small trigonometric sum with frequencies at $\mathrm{spt}(\rho)$ to destroy 4 intersection points. $\square$

Related works:

T.L.Malevich (1973): C^2-smooth Gaussian functions F on $\mathbb{R}^2$ with positive covariance function $\mathcal{E}\{F(u)F(v)\} = k(u-v) > 0$. She proved that $0 < c \le \mathcal{E}N(R;F)/R^2 \le C < \infty$.

E.Bogomolny and C.Schmit (2002): bond percolation model for description of the zero set of $2D$ Gaussian monochromatic waves (transl.-invar. Gaussian function F on $\mathbb{R}^2$ whose spectral measure is a Lebesgue measure on the unit circumference).

Challenge: "hidden universality law" that provides the rigorous foundation for the B-S work

Related works: (continuation)

F.Nazarov, M.S. (2007): 2D Gaussian monochromatic waves. There exists a constant $\nu > 0$ such that, for every $\epsilon > 0$,

$$\mathcal{P}\left\{\left|\frac{N(R;F)}{R^2} - \nu\right| > \epsilon\right\} \le C_\epsilon e^{-c_\epsilon R}, \qquad R \ge 1$$

The proof relies on Gaussian isoperimetry (Sudakov-Tsirelson, Borell).

Unfortunately, we cannot adapt the proof to other Gaussian functions.

Part II: Steps in the proof of Theorem I

Step 1: Some integral geometry

Notation: $N(u, r; F)$ the number of connected components of $Z(F)$ contained in the open ball $B(u, r)$
$\bar{N}(u, r; F)$ the number of connected components of $Z(F)$ that intersect the closed ball $B(u, r)$

Lemma: For $0 < r < R < \infty$,

$$\int_{B(R-r)} \frac{N(u,r;F)}{\mathrm{vol} B(r)}\,\mathrm{d}u \le N(R;F) \le \int_{B(R+r)} \frac{\bar{N}(u,r;F)}{\mathrm{vol} B(r)}\,\mathrm{d}u$$

We apply this with $1 \ll r \ll R$.

Notation: $(\tau_v F)(u) = F(u+v)$ (translation). Then $N(u, r; F) = N(r; \tau_u F)$

Observe: $\bar{N}(r;F) - N(r;F) \le \mathfrak{N}(r;F) \overset{\mathrm{def}}{=} \#$ of critical pts of $F\big|\partial\mathbb{B}(r)$

to be continued on the next slide

continuation

$$\frac{1-o(1)}{\mathrm{vol}B(R-r)}\int_{B(R-r)}\frac{N(r;\tau_u F)}{\mathrm{vol}B(r)}\,\mathrm{d}u \le \frac{N(R;F)}{\mathrm{vol}B(R)}$$
$$\le \frac{1+o(1)}{\mathrm{vol}B(R+r)}\int_{B(R+r)}\frac{N(r;\tau_u F)+\mathfrak{N}(r;\tau_u F)}{\mathrm{vol}B(r)}\,\mathrm{d}u, \qquad \text{for } R\gg r$$

Note: LHS and RHS are spatial averages of translations

Step 2: Metric transitivity

Fomin-Grenander-Maruyama: the spectral measure ρ has no atoms

$\Longrightarrow$ translations τ_u act metric-transitively on the Borel σ-algebra of F (that is, all invariant sets have probability 0 or 1)

$\Longrightarrow$ the distribution of $N(r;F)$ is also metric transitive

$\Longrightarrow$ Wiener's ergodic theorem can be applied

$\Longrightarrow$ for fixed r, a.s.,

$$\underline{\lim}_{R\to\infty}\frac{N(R;F)}{\mathrm{vol}B(R)} \ge \frac{\mathcal{E}N(r;F)}{\mathrm{vol}B(r)}, \qquad \overline{\lim}_{R\to\infty}\frac{N(R;F)}{\mathrm{vol}B(R)} \le \frac{\mathcal{E}N(r;F)}{\mathrm{vol}B(r)} + \frac{\mathcal{E}\mathfrak{N}(r;F)}{\mathrm{vol}B(r)}$$

Step 3: The Kac-Rice bound for the number of critical pts:

LEMMA: $\mathcal{E}\mathfrak{N}(r;F) \lesssim \mathrm{vol}_{m-1}\partial\mathbb{B}(r)$.

$\Longrightarrow$ a.s., $\lim_R \frac{N(R;F)}{\mathrm{vol}B(R)}$ exists and equals $\lim_r \frac{\mathcal{E}N(r;F)}{\mathrm{vol}B(r)} =: \nu(\rho)$

Step 4: Positivity of $\nu(\rho)$:

We already know that for fixed r, a.s., $\underline{\lim}_{R\to\infty} \frac{N(R;F)}{\mathrm{vol}B(R)} \geq \frac{\mathcal{E}N(r;F)}{\mathrm{vol}B(r)}$

$\Longrightarrow$ need to show: for some $r_0 > 0$, $\mathcal{E}N(r_0;F) > 0$.

LEMMA ON GAUSSIAN PROCESSES: Suppose μ is a compactly supported measure with $\mathrm{spt}(\mu) \subset \mathrm{spt}(\rho)$. Then for each ball $B \subset \mathbb{R}^m$ and each $\epsilon > 0$, $\mathcal{P}\{\|F - \widehat{\mu}\|_{C(\bar{B})} < \epsilon\} > 0$.

By assumption $(*)$, $\exists$ such a measure μ with $Z(\widehat{\mu})$ having a bounded connected component. By real analyticity of $\widehat{\mu}$, this component is isolated.

Choosing ϵ small enough, we get $\mathcal{P}\{N(r_0;F) > 0\} > 0$ for some r_0

$\Longrightarrow$ $\mathcal{E}N(r_0;F) > 0$. $\square$

What we cannot do:

Question: Find statistics of large components of the zero set of F; i.e., components of diameter comparable to R^α with $0 < \alpha < 1$.

Question: Find asymptotic of the variance of $N(R; F)$

Question: Prove exponential concentration of $N(R; F)/R^m$ around $\nu(\rho)$.

The difficulty is caused by components of small diameter, which do not exist when ρ is supported by a sphere $\Longrightarrow$ F satisfies the Helmholtz equation $\Delta F + \kappa^2 F = 0$

Even for Gaussian processes on $\mathbb{R}$, the question about exponential concentration remains open; cf. Tsirelson's lecture notes, Fall 2010
`http://www.tau.ac.il/~tsirel/Courses/Gauss3/main.html`

Part III: Ensembles of Gaussian real-valued "polynomials" of large degree ("Riemannian case")

X a smooth compact m-dimensional Riemannian manifold without boundary.

$\mathcal{H}_L$ a family of real finite-dimensional Hilbert spaces of smooth functions on X, $\dim\mathcal{H}_L \to \infty$ as $L \to \infty$.

$K_L(x, y)$ the reproducing kernel of the space $\mathcal{H}_L$:

$$f(y) = \langle f(\,.\,), K_L(\,.\,, y)\rangle_{\mathcal{H}_L}, \qquad f \in \mathcal{H}_L, \quad y \in X\,.$$

<u>We assume</u>: the function $x \mapsto K_L(x, x)$ does not vanish on X.

Gaussian functions on X:

The space $\mathcal{H}_L$ generates a random Gaussian function

$$f_L(x) = \sum \xi_k e_k(x), \qquad x \in X\,,$$

$\{e_k\}$ is an orthonormal basis in $\mathcal{H}_L$
ξ_k are independent standard Gaussian random variables

The covariance of the Gaussian function f_L:

$$\mathcal{E}\{f_L(x) f_L(y)\} = \sum e_k(x) e_k(y) = K_L(x,y)$$

The distribution of f_L does not depend on the choice of the orthonormal basis $\{e_k\}$ in $\mathcal{H}_L$.

Normalization:

Wlog, we assume that the functions f_L are *normalized*, that is, $\mathcal{E} f_L^2(x) = K_L(x,x) = 1$, $x \in X$.

Otherwise, replace the functions f_L and the kernel K_L by

$$\widehat{f}_L(x) = \frac{f_L(x)}{\sqrt{\mathcal{E} f_L^2(x)}}, \quad \widehat{K}_L(x,y) = \frac{K_L(x,y)}{\sqrt{K_L(x,x) \cdot K_L(y,y)}}\,.$$

This normalization changes the Hilbert spaces $\mathcal{H}_L$ but the zero sets of the Gaussian functions f_L and $\widehat{f}_L$ remain the same.

In basic examples, the function $x \mapsto K_L(x,x)$ is constant (that is, the norm of the point evaluation in $\mathcal{H}_L$ does not depend on the point), so the normalization boils down to computation of that constant.

Scaling (blowing up with the coefficient $L \gg 1$):

Notation:

$\Phi_x = \exp_x \circ I_x \colon \mathbb{R}^m \to X$, $\Phi_x(0) = x$

$\exp_x \colon T_x X \to X$ exponential map

$I_x \colon \mathbb{R}^m \to T_x(X)$ a linear Euclidean isometry

To scale the covariance kernel K_L at $x \in X$ in L times, put

$$K_{x,L}(u,v) \stackrel{\text{def}}{=} K_L\left(\Phi_x(L^{-1}u), \Phi_x(L^{-1}v)\right), \qquad u, v \in \mathbb{R}^m .$$

Note: $K_{x,L}(u,v)$ are covariance kernels of scaled Gaussian functions $f_{x,L}(u) \stackrel{\text{def}}{=} f_L(\Phi_x(L^{-1}u))$, $u \in \mathbb{R}^m$, that is,

$$K_{x,L}(u,v) = \mathcal{E}\{f_{x,L}(u) f_{x,L}(v)\}$$

Translation-invariant local limits:

DEFINITION: The Gaussian ensemble (f_L) has *translation-invariant local limits* as $L \to \infty$ if for a.e. $x \in X$, there exists a Hermitean positive definite function $k_x \colon \mathbb{R}^m \to \mathbb{R}^1$, such that for each $R < \infty$,

$$\lim_{L\to\infty} \sup_{|u|,|v|\le R} |K_{x,L}(u,v) - k_x(u-v)| = 0\,.$$

The limiting kernels $k_x(u-v)$ are covariance kernels of translation-invariant Gaussian functions $F_x \colon \mathbb{R}^m \to \mathbb{R}^1$ $(x \in X)$

$k_x = \widehat{\rho}_x, \quad \rho_x$ are prob. meas ρ_x on $\mathbb{R}^m$, symmetric w.r.t. the origin

We call the function F_x *the local limiting function*, and the measure ρ_x *the local limiting spectral measure* of the family f_L at the point x.

Note: ρ_x does not depend on the choice of the linear Euclidean isometry $I_x \colon \mathbb{R}^m \to T_x X$

Technical assumptions:

C^2-smoothness: the Gaussian ensemble (f_L) is C^2-*smooth* if *for a.e.* $x \in X$ *and for every* $R < \infty$

$$\overline{\lim_{L\to\infty}}\, \mathcal{E}\|f_{x,L}\|_{C^2(\bar{B}(R))} < \infty\,.$$

E.g., holds when ${}^{\exists}p > 4$ s.t. ${}^{\forall}R < \infty$ $\quad \overline{\lim_{L\to\infty}}\, \|K_{x,L}\|_{C^p(\bar{B}(R)\times\bar{B}(R))} < \infty$

Uniform non-degeneracy: *for a.e.* $x \in X$ *and for every* $R < \infty$

$$\underline{\lim_{L\to\infty}}\ \inf_{\bar{B}(R)}\ \det \mathrm{Cov}\big(\nabla f_{x,L}, \nabla f_{x,L}\big) > 0$$

If the limiting spectral measure is unique (i.e., ρ_x does not depend on $x \in X$) this says that the measure ρ is not supported by a hyperplane.

Notation: $N(f_L)$ the number of components of the zero set $Z(f_L)$
$\nu(\rho)$ the limiting constant from Thm I, $\bar{\nu}(x) = \nu(\rho_x)$, $x \in X$.

Remark: The measure $\bar{\nu}\,\mathrm{dvol}$ does not depend on the choice of the Riemannian metric, only the smooth structure on X matters.

THEOREM II: *Suppose that (f_L) is a C^2-smooth Gaussian ensemble on X, which has translation-invariant local limits a.e. on X. Suppose that the local limiting spectral measures ρ_x have no atoms and satisfy the non-degeneracy condition from the previous slide.*

Then the function $\bar{\nu} \in L^\infty(X)$, and

$$\lim_{L\to\infty} \mathcal{E}\left\{ \left| L^{-m} N(f_L) - \int_X \bar{\nu}\,\mathrm{dvol} \right| \right\} = 0\,.$$

Local version of Theorem II:

The value $\bar{\nu}(x) = \nu(\rho_x)$ can be recovered by a double-scaling limit:

for a.e. $x \in X$ and for each $\epsilon > 0$,

$$\lim_{R\to\infty} \lim_{L\to\infty} \mathcal{P}\left\{ \left| \frac{1}{\mathrm{vol} B(R)} N\left(x, \tfrac{R}{L}; f_L\right) - \bar{\nu}(x) \right| > \epsilon \right\} = 0$$

$N\left(x, \frac{R}{L}; f_L\right)$ is a number of connected components of the zero set $Z(f_L)$ contained in the open ball centered at x of radius R/L,
$\mathrm{vol} B(R)$ is the Euclidean volume of the ball of radius R in $\mathbb{R}^m$.

Theorem II is "an integrated version" of the local result.

Part IV: Examples to Theorem II

1. Trigonometric ensemble $X = \mathbb{T}^m$ (m-dim torus)

$\mathcal{H}_{n,m} \subset L^2(\mathbb{T}^m)$, subspace of trigonometric polynomials in m variables of degree $\le n$ in each of the variables.

The repro-kernel (= covariance): the Dirichlet kernel

$$K_{n,m}(x, y) = \prod_{j=1}^{m} \frac{\sin\left[\pi(2n+1)(x_j - y_j)\right]}{(2n+1)\sin\left[\pi(x_j - y_j)\right]}$$

scaling parameter $L = n$ (the degree).

After scaling, covariance converges together with partial derivatives of any order to the limiting kernel $k(u - v)$, $\quad k(u) = \prod_{j=1}^{m} \frac{\sin 2\pi u_j}{2\pi u_j}$.

Limiting spectral measure ρ = Lebesgue measure on the unit cube in $\mathbb{R}^m$

2. Spherical ensemble: $X = \mathbb{S}^m$ (m-dim sphere)

$\mathcal{H}_{n,m} \subset L^2(\mathbb{S}^m)$ subspace spanned by polynomials in $m+1$ variables of degree $\leq n$, restricted on $\mathbb{S}^m$.

Repro-kernel in $\mathcal{H}_{n,m}$: $c(n,m) P_n^{(\frac{m}{2}, \frac{m}{2}-1)}(x \cdot y)$, $x, y \in \mathbb{S}^m$;
$P_n^{(\alpha,\beta)}$ Jacobi polynomials of degree n and of index (α, β); i.e., polynomials orthogonal on $[-1,1]$ with the weight $(1-x)^\alpha (1+x)^\beta$.

Mehler-Heine asymptotics: $\lim\limits_{n\to\infty} n^{-\frac{m}{2}} P_n^{(\frac{m}{2}, \frac{m}{2}-1)}\big(\cos \frac{z}{n}\big) = \big(\frac{z}{2}\big)^{-\frac{m}{2}} J_{\frac{m}{2}}(z)$,
$J_{\frac{m}{2}}(z)$ is Bessel's function, the convergence is locally uniform in $\mathbb{C}$.

Scaling parameter $L = n$ (the degree)

Limiting spectral measure $\rho =$ Lebesgue measure on the unit ball in $\mathbb{R}^m$.

3. Kostlan ensemble: $X = \mathbb{PR}^m$ (m-dim projective space).

Start with homogeneous polynomials of degree n in $m+1$ variables. Their zero sets are viewed as hypersurfaces on $\mathbb{PR}^m$

The scalar product $\langle f, g\rangle = \sum_{|J|=n} \binom{n}{J} f_J g_J$, where

$$f(X) = \sum_{|J|=n} f_J X^J, \quad g(X) = \sum_{|J|=n} g_J x^J, \qquad X^J = x_0^{j_0} x_1^{j_1} x_2^{j_2} \dots x_m^{j_m},$$

$J = (j_0, j_1, j_2, \dots j_m)$, $|J| = j_0 + j_1 + j_2 + \dots + j_m$, $\binom{n}{J} = \frac{n!}{j_0! j_1! j_2! \dots j_m!}$.

Complexification: after continuation of the homogeneous polynomials f and g to $\mathbb{C}^{m+1}$, the scalar product coincides with the one in the Fock-Bargmann space $\langle f, g\rangle = c_m \int_{\mathbb{C}^{m+1}} f(Z)\overline{g(Z)} e^{-|Z|^2} \,\mathrm{dvol}(Z)$.

This is *the only unitary invariant* Gaussian ensemble of homogeneous polynomials.

Kostlan ensemble (continuation):

In homogeneous coordinates, the covariance kernel equals $\left(\frac{X\cdot Y}{|X|\,|Y|}\right)^n$. In the chart $x_0 = y_0 = 1$, we get $\left(\frac{1+(x\cdot y)}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}\right)^n$.

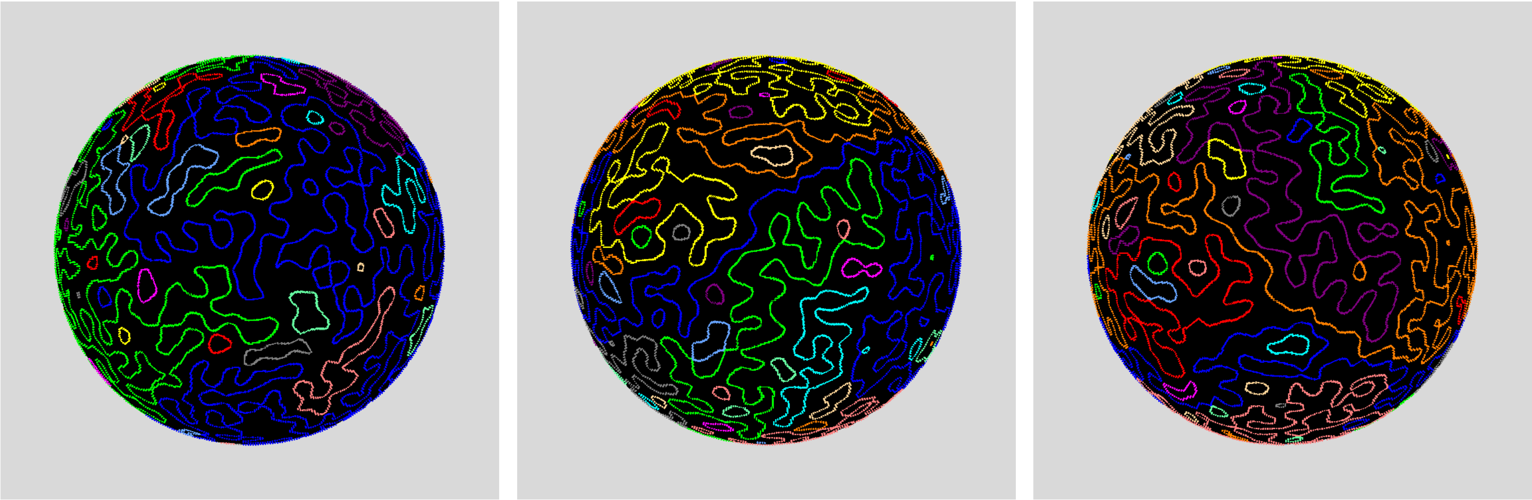
The features:

- $L = \sqrt{n}$ (square root of the degree, not the degree, as in previous examples)
- very rapid decay of the covariance away from the diagonal.

The limiting spectral measure is the Gaussian measure on $\mathbb{R}^m$ with the density $\exp\left[(x\cdot y) - \frac{1}{2}|x|^2 - \frac{1}{2}|y|^2\right]$. Once again, Theorem II is applicable.

Asymptotic distribution of of the number of components in Kostlan ensemble was recently studied by D.Gayet and J-Y.Welschinger, and by P.Sarnak and I.Wigman.

Nodal lines of Kostlan ensemble of degree 56 on $\mathbb{S}^2$



Pictures created by Maria Nastasescu

The End